MATHEMATICS FOR ALL AND FOREVER

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Abstract

This article briefly summarizes the journey of mathematics. The subject is expanding at a fast rate and it sometimes makes it essential to look back into the history of this marvelous subject. The pillars of this subject and their contributions have been briefly studied here. Since early civilization, mathematics has helped mankind solve very complicated problems. Mathematics has been a common language which has united mankind. Mathematics has been the heart of our education system right from the school level. Creating interest in this subject and making it friendlier to students' right from early ages is essential. Understanding the subject as well as its history are both equally important. This article briefly discusses the ancient, the medieval, and the present age of mathematics and some notable mathematicians who belonged to these periods.

1. Introduction

Mathematics is the abstract study of different areas that include, but not limited to, numbers, quantity, space, structure, and change. In other words, it is the science of structure, order, and relation that has evolved from elemental practices of counting, measuring, and describing the shapes of objects. Mathematicians seek out patterns and formulate new conjectures. They resolve the truth or falsity of conjectures by mathematical proofs, which are arguments sufficient to convince other mathematicians of their validity. Mathematics deals with logical reasoning and quantitative calculation, and its development has involved an increasing degree of idealization and abstraction of its subject matter.

Since the beginning of recorded history of mankind, mathematical discovery has been at the vanguard of every civilized society, and was used in even the most primitive of cultures. The needs of mathematics arose from the needs of society; the more complex a society, the more complex the mathematical needs. Primitive tribes needed little more than the ability to count, but also relied on mathematics to calculate the position of the sun and the physics of hunting. Since the 17th century CE, mathematics has been an indispensable associate to the physical sciences and technology, and more recently it has assumed a similar role in the quantitative aspects of the biological and social sciences.

1.1 Indispensible Mathematics

According to Immanuel Kant (CE 1724-1804), the celebrated German philosopher and a central figure of modern philosophy, “Mathematics is the indispensible instrument of all physical researches.” The famous German mathematician, astronomer, and physicist Johann Carl Friedrich Gauss (CE 1777-1855), known as the “prince of mathematics,” said, “Mathematics is the queen of sciences and arithmetic is the queen of mathematics.” He further added, “God does arithmetic.” Indeed God does arithmetic and we can see the evidence of this in every nook and corner of the nature he has created – starting from the hexagonal arrangement of bee hives in order to maximize space utilization to the use of Fibonacci sequence and Golden Ratio in many of the structures, both living and non-living, that He has created. With the passage of time, the queen mathematics became “the mother of science and through science we (mankind) developed.”

Numerous mathematical concepts / laws are based on ideals, and apply to an abstract, perfect world. This perfect world of mathematics is reflected in the imperfect physical world – the physical world is rarely perfect. Euclid of Alexandria, Egypt (mid-4th century - mid-3rd century BCE), the most prominent mathematician of antiquity, said, "The laws of nature are but the mathematical thoughts of God." According to Paul Adrien Maurice Dirac (CE 1902-1984), the renowned Swiss-British mathematician and Nobel Laureate in Physics, “God used beautiful mathematics in creating the world.” Galileo Galilei (1564-1642 CE), the famous Italian mathematician, astronomer, and physicist said, "Mathematics is the language with which God has written the universe.” “(The universe) cannot be read until we have learned the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles, and other geometrical figures, without which means it is humanly impossible to comprehend a single word. Without these, one is wandering about in a dark labyrinth (a place that has many confusing paths or passages through which it is difficult to find one's way or to reach the exit)." As has been said by Roger Bacon (CE 1214-1292), the English scholar, “Mathematics is the door and the key to the sciences.” The earliest American scientist Benjamin Peirce (1809-1880 CE) called mathematics, "the science that draws necessary conclusions."

Mathematics might seem an ugly and irrelevant subject at high school, but it is ultimately the study of truth – and truth is beauty! One might be surprised to find that mathematics is present in everything in nature, and everywhere in the universe, in almost every facet of life – in nature all around us, and in the technologies in our hands, although we often fail to realize it. Even those suffering from mathematics-related anxieties or phobias cannot escape its everyday presence in their lives. From home to school or to office and to all the other places we visit in-between, mathematics is ubiquitous. It has been correctly pointed out that, “Like the crest of a peacock, so is mathematics at the head of all knowledge.” We seldom note it. Looking at shape and symmetry in nature is fascinating. We enjoy looking at the nature but are not interested in going deep about what mathematical idea is stored in it. A famous Indian mathematician, Acarya Mahavira (599-527 BCE) writes that: “Whatever there is in all three worlds, which are possessed of moving and non-moving being all that indeed cannot exist as apart from Mathematics.” Some properties of mathematics like cones, spirals, hexagons that are depicted in nature, can be found everywhere if we just keep our eyes opens. As we discover more and more about our environment and our surroundings, we see that nature can be described mathematically. (Livio, 2010) Mathematics is the lingua franca of science and engineering – describing our understanding of all that we observe (and providing the foundations for many of the greatest discoveries and innovations of the last century, which remains true today as it plays a fundamental role in information and communication technologies).

Mathematics discovers, does not invent; but, it is the mother of invention – the mathematician does not act to create, but to only discover.

Mathematics is a fascinating subject; it is dynamic and rapidly developing across a wide spectrum of research areas. Many people find a sufficient reason to learn mathematics because of its coherence and elegance, and others are motivated by the fact that it plays a key role in the development of science. Indeed knowledge in mathematics develops clear logical thinking. It helps us separate out the key points in a problem and teaches how to use them to solve it. Mathematics plays an important role in developing the models that explain the physical characteristics of the universe. It is basic to the study of engineering, but its role in the social sciences and economics is not less important. Mathematics is also being increasingly used in models for biology and medicine.

1.2. The Miracle of Mathematics

Mathematics studied for its own interest is known as *pure mathematics*, which means the branches of mathematics that study and develop the principles of mathematics for their own sake, with no consideration given as to the utility of the results for practical purposes. Pure mathematics studies entirely abstract concepts. From the 18th century onwards, this was a recognized category of mathematical activity, sometimes characterized as speculative (conjectural or theoretical) mathematics, and at variance with the trend towards meeting the needs of navigation, astronomy, physics, economics, engineering, and several other areas.

One attraction of pure mathematics is that even some of the most theoretical and starkly abstract ideas that mathematicians have 'created' have turned out to describe some very physical and tangible aspects of our world. For instance, when the Greek mathematician Apollonius of Perga (262-190 BCE) studied the curves called conic sections, he wrote that he pursued his study not because his results might be helpful in certain applications, but rather because "the subject is one of those which seem worthy of study for their own sake." Yet, even though Apollonius saw that his abstract research might have some practical use, he had no idea that that nearly 2,000 years later his subject would be used to describe with precision how the planets in our solar system orbit around the Sun. (Smith, 2013) When the celebrated French mathematician, scientist, and philosopher Rene Descartes (1596-1650 CE) created the coordinate system of geometry, which freed mathematicians to compute in multiple hypothetical dimensions beyond the three experienced in daily lives, he had no idea that six-dimensional geometry would one day be found perfectly reflected in the 'waggle dance' which the honeybees use to tell other bees about the distance, direction, and quality of food sources from their hive where they have found food. And when scholars such as Leonhard Euler (CE 1707-1783), a pioneering Swiss mathematician and physicist, and Johann Carl Friedrich Gauss (CE 1777-1855), Germany's greatest mathematician and physicist, explored the mathematically absurd idea of 'imaginary' square roots of negative numbers, they would never have conceived that their 'imaginary numbers' would form an integral part of the most fundamental laws of physics known in our time: quantum mechanics, a theory of the mechanics of atoms, molecules, and other physical systems which are subject to the uncertainty principle. (Smith, 2013) (Quantum mechanics suggests that the behavior of matter and energy is inherently probabilistic and that the effect of the observer on the physical system being observed must be understood as a part of that system.)

The Miracle of Mathematics

"So, how is it that even pure mathematics, developed, explored, and extended over centuries with no apparent connection to reality, is sometimes found to be not abstract at all — but, rather, to be part of the very fabric of our very real, very concrete universe?" (Smith, 2013) Some have called this phenomenon the "miracle of mathematics." Nobel Laureate Eugene Paul Wigner (CE 1902-1995), the Hungarian-American theoretical physicist and mathematician, wrote that "the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it." (Wigner, 1960) He observed that: "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve." He further noted that the mathematical structure of a physical theory often points the way to further advances in that theory and even to empirical predictions. (Wigner, 1960)

"Natural science alone cannot explain why we find, beneath the reality around us, such a beautiful, ordered, and systematic foundation of mathematics. It is an astonishing fact for which scientists have no natural explanation, though many have tried to provide one. It remains a fundamental mystery of science." (Smith, 2013) "Even if there is only one possible unified theory, it is just a set of rules and equations. What is it that breathes fire into the equations and makes a universe for them to describe?" (Hawking, 1998)

Beginning his paper with the belief (which is common to all those familiar with mathematics) that mathematical concepts have applicability far beyond the context in which they were originally developed, and based on his experience, Wigner (1960) wrote: "it is important to point out that the mathematical formulation of the physicist's often crude experience leads in an uncanny number of cases to an amazingly accurate description of a large class of phenomena." As an example, he then cited the fundamental law of gravitation, originally used to model freely falling bodies on the surface of the earth, which was extended on the basis of what Wigner terms "very scanty observations" to describe the motion of the planets, where it "has proved accurate beyond all reasonable expectations."

The immense usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it. "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning." (Wigner, 1960)

2. VOYAGE OF MATHEMATICS

The word 'mathematics' itself comes from the Greek word 'mathema' meaning science, knowledge, or learning, and in modern Greek just 'lesson' (or 'subject of instruction'). The earliest records of mathematics show it arising in response to practical needs in agriculture, business, and manufacturing. At the dawn of civilization, man developed two important ideas, viz., multiplicity and space. The first concept involved counting (of animals, days, etc.) and the second involved areas and volumes (of land, water, crop yield, etc.), which in turn evolved into two important branches of mathematics, viz., arithmetic (comes from the Greek word arithmetike, a compound of the words arithmos for 'number' and techne for 'art' or 'skill') and geometry (derived from the Greek roots 'geo' meaning 'earth' and 'metrein' meaning 'measure').

VOYAGE OF MATHEMATICS

In Egypt and Mesopotamia (a region of western Asia, in what is now Iraq, known as the 'cradle of civilization'), where evidence dates from the 2nd and 3rd millennia BCE, mathematics was used for many practical applications, ranging from surveying fields after the annual floods to making the complex calculations necessary for building the pyramids. Estimates of the value of π (pi) were found in both these locations. By about 3000 BCE, mathematicians of ancient Egypt used an additive base-ten system that was without place values. They also developed geometric formulas for finding the area and volume of simple figures.

This early mathematics was essentially empirical in nature, arrived at by trial and error as the best available means for obtaining results, with no proofs given. However, it is now known that the Babylonians were aware of the necessity of proofs prior to the Greeks, who had been presumed the originators of this important step. (Babylonia was an ancient Semitic nation state and cultural region based in central-southern Mesopotamia). By 2100 BCE, Babylonians had developed a sexagecimal (base-60) numeral system, which had important uses in astronomy and also in commerce, as sixty can be divided easily (60 has many divisors), and works well with a calendar. The system survives today in the way we measure time and angles. From this derives the modern day usage of 60 seconds in a minute, 60 minutes in an hour, and 360 (60 x 6) degrees in a circle, as well as the use of seconds and minutes of arc to denote fractions of a degree. The Babylonians also had a place-value system to represent numbers of any size — digits written in the left column represented larger values, much as in the decimal system. However, they lacked an equivalent of the decimal point, and so the place value of a symbol often had to be inferred from the context. On the other hand, this 'defect' is equivalent to the modern-day usage of floating point arithmetic. (Floating point arithmetic derives its name from something what happens when one uses exponential notation. The number 123 can be written using exponential notation as: 1.23×10^2, 12.3×10^1, 123×10^0, $.123 \times 10^3$, 1230×10^{-1}. It may be noted how the decimal point 'floats' within the number as the exponent is changed. This phenomenon gives floating point numbers their name.)

Rhind Mathematical Papyrus or simply *Rhind Papyrus* (a famous document, and the major source of our knowledge of ancient Egyptian mathematics, that dates to around 1650 BCE, housed in the British Museum since 1864 CE, and first published in 1879 CE), named after Alexander Henry Rhind, a Scottish antiquarian who purchased the *papyrus* in 1858 CE in Luxor, Egypt, is the most extensive Egyptian mathematical text, an instruction manual for students in arithmetic and geometry. Besides giving formulas for computing areas and methods for multiplication, division, and working with unit fractions, *Rhind Papyrus* also provides evidence of mathematical knowledge such as composite and prime numbers; arithmetic, geometric, and harmonic means; and simplistic understandings of both the *Sieve of Eratosthenes* and *perfect number theory* (namely, that of the number 6). It also shows how to sum arithmetic and geometric series and solve first order linear equations.

2.1 Greek Mathematics

Greek mathematics developed around the Eastern shores of the Mediterranean from the 7th century BCE to the 4th century CE. A profound change occurred in the nature and approach to mathematics with the contributions of the Greek mathematicians who lived in cities spread over the entire Eastern Mediterranean, from Italy to North Africa, but were united by culture and language. Greek mathematics of the period following Alexander the Great is sometimes called Hellenistic (or Hellenic) mathematics, which is represented by Thales, Pythagoras, Plato, and Aristotle, and by the schools associated with them. The *Pythagorean Theorem*, known earlier in Mesopotamia, was discovered by the Greeks during this period.

Historians traditionally place the beginning of Greek mathematics to the age of Thales of Miletus (624-546 BCE), who is credited for giving some logical reasoning for several elementary results involving circles and angles of triangles rather than basing them on intuition and experimentation. The two earliest mathematical theorems, viz., *Thales' theorem* and *Intercept theorem*, are attributed to him.

Thales' theorem states that an angle inscribed in a semicircle is a right angle, which might have been learned by Thales while in Babylon but tradition attributes to Thales a demonstration of the theorem and it is for this reason that Thales is often hailed as the father of the deductive organization of mathematics and as the first true mathematician. But, it is not known whether or not Thales was the one who introduced into mathematics the logical structure that is so widespread today; it is, however, known that within two hundred years of Thales the Greeks had introduced logical structure and the idea of proof into mathematics.

Thales went to Egypt where he studied with the priests and learned of mathematical discoveries, the knowledge of which he brought back to Greece. He is known for both theoretical and practical understanding of geometry and did geometrical research. Using triangles, Thales applied his knowledge of geometry to estimate the distance of ships at sea from the shore, which was particularly important to the Greeks, because they wanted to know whether the ships were coming for trade or for battle. While in Egypt, Thales is supposed to have determined the height of a pyramid by measuring the length of its shadow when the length of his own shadow was equal to his height. He is thought to be the earliest known man in history to whom specific mathematical discoveries have been attributed. It is, however, possible that Thales has been given credit for discoveries that were not really his. Thales advised Anaximander's student, Pythagoras, to visit Egypt in order to continue his studies in mathematics and philosophy.

Another important figure in the development of Greek mathematics is Pythagoras of Samos (572-495 BCE), who, like Thales, also traveled to Egypt and Babylon, then under the rule of Nebuchadnezzar, but settled in Croton, Magna Graecia (ancient Greek colonies of Southern Italy). Pythagoras founded a scholarly society (with secret rites) called the *Pythagorean Brotherhood* (an academy for the study of philosophy, mathematics, and natural science). In other words, he established an order known as the *Pythagoreans*, which held knowledge and property in common and hence all of the discoveries by individual Pythagoreans were attributed to the order.

And since in antiquity it was customary to give all credit to the master, Pythagoras himself was given credit for the discoveries made by his order. (Aristotle, however, refused to attribute anything specifically to Pythagoras as an individual and only discussed the work of the Pythagoreans as a group.) One of the most important characteristics of the Pythagorean order was that the pursuit of philosophical and mathematical studies was a moral basis for the conduct of life. The *Pythagorean Brotherhood* and a handful of other mathematicians of ancient Greece introduced a more rigorous mathematics than what had been earlier, building from first principles using axioms and logic. For example, geometry before Pythagoras was merely a collection of rules derived by empirical measurement. Pythagoras discovered that a complete system of mathematics could be constructed, where geometric elements corresponded with numbers, and where integers and their ratios were all that was necessary to establish an entire system of logic and truth. Pythagoras is mainly remembered for his most important result what is known as *Pythagoras' Theorem* (or the *Pythagorean Theorem* known earlier in Mesopotamia and discovered by the Greeks during this period): that, for any right-angled triangle, the square of the length of the hypotenuse is equal to the sum of the square of the other two sides (written in the form of an equation as $a^2 + b^2 = c^2$). What Pythagoras and his followers did not realize is that this also works for any shape, i.e., the area of a pentagon on the hypotenuse is equal to the sum of the pentagons on the other two sides, as it does for a semi-circle or any other regular (or even irregular) shape. Another important result due to Pythagoras was the *irrational quantities* (which struck a blow against the supremacy of the whole numbers).

For example, the Pythagorean, Hippasus, is credited with the discovery that the side of a square and its diagonal are incommensurable, i.e., a square of length 1 has a diagonal of irrational length. It has been customarily said that the Pythagoreans discovered most of the material in the first two books of Euclid's *Elements*. The words 'philosophy' (love of wisdom) and 'mathematics' (that which is learned) are said to have been coined by Pythagoras. Indeed from this love of knowledge came many achievements.

Plato (427-347 BCE) studied Philosophy in Athens under Socrates and then moved to Cyrene (a city of Libya) in North Africa where he studied mathematics under Theodorus. On his return to Athens and drawing inspiration from Pythagorus, Plato founded an academy, where he stressed mathematics as a way of understanding more about reality. His influence on mathematics was not due to any mathematical discoveries, but rather to his conviction that the study of mathematics provides the best training for the mind, and was hence essential for the cultivation of philosophers. The renowned motto over the door of his academy read: "Let no one ignorant of geometry enter here." He was confident that geometry was the key to unfolding the secrets of the universe. Plato is perhaps best known for identifying 5 regular symmetrical 3-dimensional shapes, known as the *Platonic Solids* (which he claimed to be the basis for the whole universe): the tetrahedron (constructed of 4 regular triangles, representing fire), the octahedron (composed of 8 triangles, representing air), the icosahedron (composed of 20 triangles, representing water), the cube (composed of 6 squares, representing earth), and the dodecahedron (made up of 12 pentagons, which Plato obscurely described as "the god used for arranging the constellations on the whole heaven").

Greek Mathematics

Aristotle (384-322 BCE), a student of Plato’s academy in Athens, was not primarily a mathematician but made important contributions by systematizing deductive logic. His contribution to mathematics is his analysis of the roles of definitions and hypotheses in mathematics. He wrote on physical subjects: some parts of his *Analytica Posteriora* (The Posterior Analytics) show an unusual grasp of the mathematical method. Another student of Plato’s academy, Eudoxus (408-355 BCE) made important contributions to the *theory of proportion*, where he made a definition allowing possibly irrational lengths to be compared in a similar way to the method of cross multiplication used today. His contributions to the early theory of proportions (equal ratios) forms the basis for the general account of proportions found in Book V of Euclid’s *Elements* (300 BCE). Another important contribution of Plato is his *theory of exhaustion*, considered the geometric forerunner of the modern notion of ‘limit’ in integral calculus, found application in determination of areas and volumes of sophisticated geometric figures. This process was used by Archimedes to determine the area of the circle.

After Alexander entered Egypt and established the city of Alexandria at the mouth of the Nile in 332 BCE, Alexandria became the intellectual metropolis of the Greek race until its fall in 641 CE in the hands of Arabs. Among the scholars attracted to Alexandria around 300 BCE was Euclid, who set up a school of mathematics. He probably attended Plato's academy in Athens before moving to Alexandria. Euclid’s genius was not so much in creating new mathematics but rather in the presentation of old mathematics in a clear, logical, and organized manner, providing an axiomatic development of the subject. Euclid gathered up all of the knowledge developed in Greek mathematics at that time and created his great book the *Stoicheion* or *Elements* (300 BCE), which is divided into 13 volumes and contains 465 propositions from plane and solid geometry, and from number theory. (The *method of exhaustion* is a technique for finding the area of a shape by inscribing within it a sequence of polygons whose areas converge to the area of the containing shape. If the sequence is correctly constructed, the difference in area between the n^{th} inscribed polygon and that of the containing shape will become arbitrarily small as n becomes large. As the space between the inscribed polygon and the containing shape becomes arbitrarily small, then the possible values for the area of the containing shape are systematically ‘exhausted’ by the lower bound polygonal areas successively established by the sequence members.) Archimedes used his *method of exhaustion* for computing areas of curvilinear plane figures, volumes bounded by curved surfaces. In fact, his most sophisticated use of the *method of exhaustion*, which remained unsurpassed until the development of integral calculus in the 17^{th} century, was his proof — known as the *Quadrature of the Parabola* — that the area of a parabolic segment (enclosed by a parabola and a straight line) is 4⁄3 that of a certain inscribed triangle. He dissected the area into infinitely many triangles whose areas form a geometric progression. He then computed the sum of the resulting geometric series, and proved that this is the area of the parabolic segment. By the *method of exhaustion*, he effectively approximated π; his estimate was between 3.1429 and 3.1408, which compares well with its actual value of approximately 3.1416. His contributions in geometry revolutionized the subject and his methods anticipated the integral calculus 2,000 years before Newton and Leibniz; he was able to use infinitesimals in a way that is similar to modern integral calculus.

Besides pure mathematics, Archimedes also made significant contributions to applied mathematics as well, especially fluid mechanics. He solved the problem of buoyancy, established the law of the lever, defined the spiral, found an ingenious system for expressing large numbers, and devised a method of displacement of volumes (that an object totally or partially immersed in a fluid (liquid or gas) is buoyed (lifted) up by a force equal to the weight of the fluid that is displaced), which is known as *Archimedes' Principle* or the *Eureka formula*. He was also a thoroughly practical man who invented a wide variety of machines including pulleys and the Archimidean screw pumping device.

The Greek mathematician Nicomachus of Gerasa (then a Greek city, now Jerash, Jordan) (60-120 CE), an important mathematician in the ancient world, is best known for his works *Introduction to Arithmetic* (Arithmetike Eisagoge) and *Manual of Harmonics* (Encheiridion Harmonikes or Manuale Harmonicum) in Greek. The *Introduction to Arithmetic*, Nicomachus' most famous work, was the first work to treat arithmetic as a separate topic from geometry and continued to be an important source for the theory of number and calculation well into the medieval period. The *Introduction to Arithmetic* also contained the earliest-known Greek multiplication tables. In the front matter to his translation of the *Introduction to Arithmetic*, the Dutch-American classical scholar Martin Luther D'Ooge wrote "During his own lifetime, he enjoyed, apparently, the highest reputation as a mathematician, and after his death he continued to be studied, directly or indirectly, by generation after generation of schoolboys." Manlius Severinus Boethius' *De Institutione Arithmetica* is in large part a Latin translation of *Arithmetike Eisagoge*. He wrote extensively on numbers, especially on the significance of prime numbers and perfect numbers and argued that arithmetic is ontologically prior to the other mathematical sciences (geometry, astronomy, and music), and is their cause. *Nicomachus's Theorem* states that the n^{th} cubic number n^3 is a sum of n consecutive odd numbers, for example, $1^3 = 1$, $2^3 = 3 + 5$, $3^3 = 7 + 9 + 11$, $4^3 = 13 + 15 + 17 + 19$, etc. In general, $n^3 = (n^2 – n + 1) + (n^2 – n + 3) + … + (n^2 + n – 1)$. His two other works on mathematics, namely, *Art of Arithmetic* and *Introduction to Geometry*, are lost. Unlike Euclid, Nicomachus gave no abstract proofs of his theorems, only stated the theorems and illustrated them with numerical examples.

As a Neo-Pythagorean, Nicomachus was often more interested in the philosophical questions dealing with whole numbers, and in the mystical properties of numbers, rather than their mathematical properties. He distinguished between the wholly conceptual immaterial number, which he regarded as the *divine number*, and the numbers which measure material things, the *scientific number*. His work *Theology of Arithmetic* on the Pythagorean mystical properties of numbers is lost.

Nicomachus' *Manual of Harmonics* is the first important treatise on music theory since the time of Aristoxenus and Euclid. It provides the earliest surviving record of the story of Pythagoras' epiphany outside a smithy that pitch is determined by numeric ratios. Nicomachus also gives the first in-depth account of the relationship between music and the ordering of the universe via the 'music of the spheres'. Most Arabic texts on number theory written by mathematicians were influenced by both Euclid and Nicomachus.

Greek Mathematics

Claudius Ptolemy (about 85 – about 165 CE), an Egyptian astronomer and mathematician, of Greek descent who flourished in Alexandria, has a prominent place in the history of mathematics primarily because of the mathematical methods he applied to astronomical problems. Ptolemy wrote *Mathematike Syntaxis* (The Mathematical Compilation), a treatise on the apparent motions of the stars and planets. This work soon became known as *The Greatest Compilation* and it established the model of a geocentric universe, a scientific schema that would be followed for the next thousand years. During the 9th century CE, however, when Ptolemy's Greek work was adopted in the Islamic world, its title in Arabic was shortened to *The Greatest*, which when transliterated into Latin became *Almagest*, the name still used today. (In the 9th century CE, at least three translations of the *Almagest* into Arabic were made; the most influential one was made by the translator Ishaq ibn Hunain (809-873 CE) and revised by Thabit ibn Qurra (827-901 CE).) Among Ptolemy's less familiar books is the *Harmonics*, an attempt to apply mathematical models to the aesthetics of musical tones and tunings. In this, Ptolemy reasoned that certain kinds of ratios between the whole numbers could account for the practices of the musicians of his time.

His contributions to trigonometry are especially important. For instance, Ptolemy's table of the lengths of chords in a circle is the earliest surviving table of a trigonometric function. He also applied fundamental theorems in spherical trigonometry (apparently discovered half a century earlier by Menelaus of Alexandria) to the solution of many basic astronomical problems. In particular, he introduced a trigonometric function which he called 'the *chord function*' and denoted it by Crd, which is related to our modern sine function, by the formula: $Crd(a) = 120.sin(2a)$. The sine function, originally an Indian discovery, simplified the geometric procedures needed in mathematical astronomy. He developed and used formulas for his chord function which are analogous to the modern addition and subtraction trigonometric formulae: $\sin(a+b) = \sin(a).\cos(b) + \sin(b).\cos(a)$ and $\sin(a - b) = \sin(a).\cos(b) – \sin(b).\cos(a)$. These were of practical use and enabled him to create a table of the Crd function at intervals of half a degree.
In Euclidean geometry, Ptolemy's theorem is a relation between the four sides and two diagonals of a cyclic quadrilateral (a quadrilateral whose vertices lie on a common circle) which is stated as follows: If a quadrilateral is inscribable in a circle then the product of the measures of its diagonals is equal to the sum of the products of the measures of the pairs of opposite sides. Ptolemy used the theorem as an aid to creating his table of chords, a trigonometric table that he applied to astronomy. Moreover, the converse of Ptolemy's theorem is also true: In a quadrilateral, if the sum of the products of its two pairs of opposite sides is equal to the product of its diagonals, then the quadrilateral can be inscribed in a circle.

He obtained, using his tables on a 360-gon (that is, a regular polygon with 360 sides) inscribed in a circle, the approximation $\pi \approx 3 + 17/120 = 3.14166$, and, since $Crd(60º) = \sqrt{3}$, he also obtained from his tables, the approximation, $\sqrt{3} \approx 1.73205$.

2.2 Chinese Mathematics

China is one of the oldest civilizations, comparable only to Egypt and Babylonia, which were well-versed in mathematics. Chinese mathematics originally developed to aid record keeping, land surveying, and building. Chinese mathematics was, like their language, very concise — it was very much problem based, motivated by problems of the calendar, trade, land measurement, architecture, government records, and taxes. However, to understand about ancient Chinese mathematics, the first thing is the way in which it differs from Greek mathematics; unlike Greek mathematics, there is no axiomatic development of mathematics. The Chinese concept of mathematical proof is radically different from that of the Greeks.

By the 100's BCE, the Chinese had devised a decimal system of numbers that included fractions, zero, and negative numbers. The simple but efficient ancient Chinese numbering system, which dates back to at least the 2nd millennium BCE, used special small bamboo sticks known as *counting rods* arranged to represent the numbers 1 to 9, which were then placed in columns representing units, tens, hundreds, thousands, etc. Of particular note is the use in Chinese mathematics of a decimal positional notation system, the so-called 'rod numerals' in which distinct ciphers were used for numbers between 1 and 10, and additional ciphers for powers of ten.

Thus, the number 123 would be written using the symbol for '1', followed by the symbol for '100', then the symbol for '2' followed by the symbol for '10', followed by the symbol for '3'. It was, therefore, a decimal place value system very similar to the one used today (in fact it was the first such number system, adopted by the Chinese over a thousand years before it was adopted in the West) and it made even quite complex calculations very quick and easy. They solved arithmetic problems with the aid of counting rods. This was the most advanced number system in the world at the time, apparently in use several centuries before the Common Era (CE) and well before the development of the Indian numeral system. The Chinese also used these devices to solve equations — even groups of simultaneous equations in several unknowns. Written numbers, however, employed the slightly less efficient system of using a different symbol for tens, hundreds, thousands, etc. This was largely because there was no concept or symbol of zero, and it had the effect of limiting the usefulness of the written number in Chinese.

The main thrust of Chinese mathematics, however, developed in response to the empire's growing need for mathematically competent administrators. Possibly the most important early Chinese mathematical work is the *Jiuzhang Suanshu* (The Nine Chapters on the Mathematical Art), a handbook of practical problems that was compiled in the first two centuries (100's and 200's) BCE, probably by a variety of authors, became an important tool in the education of such a civil service, covering hundreds of problems in practical areas such as trade, taxation, engineering, and the payment of wages. (Kangshen, Crossley, and Lun, 1999) *Jiuzhang Suanshu* is the longest surviving mathematical texts from China, the first being *Suan Shu Shu* or *Book of Numbers and Computations* (202-186 BCE) and *Zhou Bi Suan Jing* or *The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven* (compiled throughout the Han dynasty, 206-220 CE). It sketched an approach to mathematics focusing on finding the most general methods of solving problems, which may be contrasted with the approach common to ancient Greek mathematicians, who worked for deducing propositions from an initial set of axioms.

Chinese Mathematics

In 263 CE, the Chinese mathematician Liu Hui wrote a commentary on *The Nine Chapters on the Mathematical Art*. Liu also wrote *Haidao suanjing* or *Sea Island Mathematical Manual* which was originally an appendix to his commentary on Chapter 9 of the *Nine Chapters*. In it, Liu used Pythagoras's theorem to calculate heights of objects and distances to objects which cannot be measured directly. This was to become one of the themes of Chinese mathematics. Liu Hui gave a more mathematical approach than earlier Chinese texts, providing principles on which his calculations are based. Among Liu Hui's greatest achievements was his analysis of a mathematical statement called the *Gou-Gu Theorem.* The theorem, known as the *Pythagorean Theorem* in the West, describes a special relationship that exists between the sides of a right triangle. Liu Hui calculated the value of π more accurately than ever before. His best approximation of π was 3.14159 which he achieved from a regular polygon of 3072 (equal) sides. It is clear that he understood iterative processes and the notion of a limit.

The beginning of Chinese algebra is seen in the work of Wang Xiaotong (about 580 - about 640 CE), who wrote the *Jigu Suanjing* or *Continuation of Ancient Mathematics*, a text with only 20 problems based mostly on engineering construction of astronomic observation tower, dike (long wall or embankment to prevent flooding from the sea), excavation of a canal bed etc and right angled triangles. *Jigu Suanjing* later became one of the *Ten Computational Canons* (or the *Ten Classics*), and in this Wang solved cubic equations by extending an algorithm for finding cube roots. His work is seen as a first step towards the 'tian yuan' or 'coefficient array method' or 'method of the celestial unknown' of Li Zhi for computing with polynomials.

Qin Jiushao (1202-1261 CE) developed a method of solving simultaneous linear equations using a method of repeated approximations. He wrote his only mathematical book *Shushu Jiuzhang* or *Mathematical Writings in Nine Sections* (1247 CE), which is divided into nine 'categories', each containing nine problems related to calendrical computations, meteorology, surveying of fields, surveying of remote objects, taxation, fortification works, construction works, military affairs, and commercial affairs. The categories are concerned with indeterminate analysis, calculation of the areas and volumes of plane and solid figures, proportions, calculation of interest, progressions, simultaneous linear equations, and solution of quadratic, cubic, and higher-degree polynomial equations in one unknown using a method of repeated approximations. Every problem is followed by a numerical example, a general solution, and a description of the calculations performed with counting rods. The two most important methods found in *Shushu Jiuzhang* are for the solution of simultaneous linear congruences: the Chinese *Remainder Theorem*, where Qin considered problems of the type $x \equiv r_k (\text{mod } m_k)$, $k = 1, 2, ..., n$ (a linear congruence is the problem of finding an integer x satisfying ax ≡ b (mod m) for specified integers a, b, and m), and an algorithm for obtaining a numerical solution of higher-degree polynomial equations based on a process of successively better approximations. This method was rediscovered in Europe about 1802 CE and is known as the *Ruffini-Horner method*. Although Qin's is the earliest surviving description of this algorithm, most scholars believe that it was widely known in China before this time.

Chinese Mathematics

Yang Hui (1238-1298 CE), a contemporary of Qin Jiushao, wrote several outstanding mathematical texts including the *Xiangjie jiuzhang suanfa* (Detailed analysis of the mathematical rules in the *Nine Chapters* and their reclassifications). He worked on magic squares, magic circles, and the binomial theorem, and is best known for his contribution of presenting 'Yang Hui's Triangle', which was the same as Pascal's Triangle up to the 6th row, discovered by Yang's predecessor Jia Xian. Yang also gave formulae for the sums of certain series, for example, he found the sum of the squares of the natural numbers from m^2 to $(m + n)^2$ and showed that $1 + 3 + 6 + ... + n(n + 1)/2 = n(n + 1)(n + 2)/6$. Yang's other important books are *Riyong suanfa* (Mathematics for everyday use, an elementary text) and *Yang Hui suanfa* (Yang Hui's methods of computation).

The high tide of Chinese mathematics occurs in the 13th century CE (latter part of the Sung period), with the further development of Chinese algebra. The most important text from that period is the *Precious Mirror of the Four Elements* (Ssu-yuan yu-chien) by Chu Shih-chieh (fl. 1280-1303 CE), dealing with the solution of simultaneous higher order algebraic equations using a method similar to *Horner's method*. The *Precious Mirror* also contains a diagram of Pascal's triangle with coefficients of binomial expansions through the 8th power, though both appeared in Chinese works as early as 1100 CE. The Chinese also made use of the complex combinatorial diagram known as the magic square and magic circles, described in ancient times and perfected by Yang Hui (1238-1298 CE). In *Ssu-yuan yu-chien*, Chinese algebra reached its peak of development, but this work also marked the end of the golden age of Chinese mathematics, which began with the works of Liu I, Chia Hsien, and others in the 11th and the 12th centuries CE, and continued in the following century with the writings of Ch'in Chiu-shao, Li Chih, Yang Hui, and Chu Shih-chieh himself.

2.3 Indian Mathematics

Despite developing quite independently of Chinese (and probably also of Babylonian mathematics), some very advanced mathematical discoveries were made at a very early time in India.

The oldest extant mathematical text from India is the *Sulba Sutras* (or *Sulva Sutras*) (dated variously between the 8th century BCE and the 2nd century CE), (Boyer, 1991a, p. 207) appendices to religious texts which give simple rules for constructing altars of various shapes, such as squares, rectangles, parallelograms, and others. (Puttaswamy, 2000) As with Egypt, the preoccupation with temple functions points to an origin of mathematics in religious ritual. (Boyer, 1991a) The *Sulba Sutras* give rules for constructing a circle with approximately the same area as a given square, which imply several different approximations of the value of π (pi). (Kulkarni, 1978) In addition, they compute the square root of 2 to several decimal places, list Pythagorean triples, and give a statement of the simplified *Pythagorean Theorem* for the sides of a square and for a rectangle. (Boyer, 1991a) The *Sutras* also contain geometric solutions of linear and quadratic equations of the form $ax^2 = c$ and $ax^2 + bx = c$ (in a single unknown), and give a notably accurate figure for the square root of 2, by adding $1 + 1/3 + 1/(3 \times 4) + 1/(3 \times 4 \times 34)$, which leads to a value of 1.4142156, correct to 5 decimal places. However, according to Carl Benjamin Boyer, an American historian of sciences, especially mathematics, “All of these results are present in Babylonian mathematics, indicating Mesopotamian influence. It is not known to what extent the *Sulba Sutras* influenced later Indian mathematicians. As in China, there is a lack of continuity in Indian mathematics; significant advances are separated by long periods of inactivity.” (Boyer, 1991c, p. 207-209)

The great Indian mathematician and astronomer Aryabhata I (476-550 CE) wrote the *Aryabhatiya* (499 CE), a slim volume, written in verse, intended to supplement the rules of calculation used in astronomy and mathematical mensuration, though with no feeling for logic or deductive methodology. (Boyer, 1991a, p. 210) It is in the *Aryabhatiya* that the decimal place-value system first appears, and in this, he provided elegant results for the summation of series of squares and cubes. Several centuries later, the Persian mathematician Abu Rayhan al-Biruni described the *Aryabhatiya* as a "mix of common pebbles and costly crystals". (Boyer, 1991a, p.211) Applying the half chord rather than the full chord method used by Greeks, Aryabhata I made the fundamental advance in finding the lengths of chords of circles. He gave the value of π (pi) as 3.1416, claiming for the first time that it was an approximation (and giving it in the form that the approximate circumference of a circle of diameter 20000 is 62832). He gave methods for finding square roots, summing arithmetic series, solving indeterminate equations of the type $ax - by = c$. He also gave what later came to be known as the table of sines, and wrote a text book for astronomical calculations, *Aryasiddhanta*. This data is used in preparing Hindu calendars (Panchangs) even today.

Varahamihira (505-587 CE) of the Ujjain school of mathematics, made some important mathematical discoveries, among which are certain trigonometric formulas which when translated into our present day notation correspond to $\sin x = \cos(\pi/2 - x)$, $\sin^2 x + \cos^2 x = 1$, and $(1 - \cos 2x)/2 = \sin^2 x$. Another important contribution to trigonometry was his sine tables where he improved those of Aryabhata I giving more accurate values. Varahamihira also considered the problem of computing ${}_nC_r$ (combination of n things taken r at a time) writing the numbers n in a column with $n = 1$ at the bottom and the numbers r in rows with $r = 1$ at the left-hand side. Starting at the bottom- left side of the array which corresponds to the values $n = 1$, $r = 1$, the values of ${}_nC_r$ are found by summing two entries, namely the one directly below the (n, r) position and the one immediately to the left of it. In spite of being seen from a different angle from the way it is developed today, Varahamihira's table is none other than *Pascal's triangle* for finding the binomial coefficients.

Brahmagupta (598-665 CE), the great ancient Indian mathematician and astronomer, is renowned for introduction of negative numbers and operations on zero into arithmetic which stood for 'nothing'. He described zero as the result of subtracting a number from itself, and established the basic mathematical rules for dealing with zero (1 + 0 = 1; 1 − 0 = 1; and 1 × 0 = 0), although his understanding of division by zero was incomplete (he thought that 1 ÷ 0 = 0). He gave rules for arithmetical operations among negative numbers ('debts') and positive numbers ('property'), as well as surds. His most famous work was *Brahmasphutasiddhanta* (Correctly Established Doctrine of Brahma), a corrected version of old astronomical treatise *Brahmasiddhanta*, was written completely in verse and does not contain any kind of mathematical notation. Besides a good understanding of the role of zero and rules for manipulating both negative and positive numbers, *Brahmasphutasiddhanta* contains ideas including a method for computing square roots, methods of solving linear and quadratic equations, and rules for summing series, *Brahmagupta's identity* (i.e., the product of two numbers of the form $a^2 + nb^2$ is itself a number of that form), and *Brahmagupta's theorem* (which states that if a cyclic quadrilateral is orthodiagonal, i.e., has perpendicular diagonals, then the perpendicular to a side from the point of intersection of the diagonals always bisects the opposite side).

Brahmagupta's works, especially the *Brahmasphutasiddhanta*, were brought by the 8th-century (CE) Abbasid caliph al-Mansur to *Bayt al-Ḥikmah* (The House of Wisdom), his newly founded centre of learning at Baghdad, providing an important link between Indian mathematics and astronomy and the promising rise of science and mathematics in the Islamic world. *Brahmasphutasiddhanta* was translated into Arabic as *Sind Hind* (CE 770) and it was from this translation that Islamic mathematicians were introduced to Indian numeral system, which they adapted as *Arabic numerals*. Islamic scholars carried knowledge of this number system to Europe by the 12th century CE, which has now displaced all older number systems throughout the world. (Boyer, 1991b, p. 226) Perhaps his most famous result was a formula for the area of a cyclic quadrilateral (i.e., quadrilateral inscribed in a circle) with sides a, b, c, and d as $\Delta = \sqrt{[(s-a)(s-b)(s-c)(s-d)]}$ where $2s = a+b+c+d$. He gave a valuable interpolation formula for computing sines. He also gave the sum of the squares of the first n natural numbers as $n(n + 1)(2n + 1)/6$ and the sum of the cubes of the first n natural numbers as $(n(n + 1)/2)^2$. He also gave the following formula, used in G.P. series: $a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} = (ar^{n-1}) \div (r-1)$. Brahmagupta gave partial solutions to certain types of indeterminate equations of the second degree with two unknown variables. (Majumdar, 1981)

The fame of Bhaskara I, (about 600 – about 680 CE), an astronomer and mathematician, who helped disseminate the mathematical work of Aryabhata I (476-550 CE), rests on three treatises he composed on the works of Aryabhata I. Two of these treatises, known today as *Mahabhaskariya* (Great Book of Bhaskara) and *Laghubhaskariya* (Small Book of Bhaskara), are astronomical works in verse, while his *Aryabhatiyabhashya* (629 CE) is a prose commentary on the *Aryabhatiya* of Aryabhata I. In his treatise the *Mahabhaskariya*, Bhaskara I included three verses which give a remarkably accurate approximation for the sine function: in modern notation, the sine approximation formula can be expressed as $\sin x = 16x(\pi - x) / [5\pi^2 - 4x(\pi - x)]$, where x is in degrees. Bhaskara explains in detail Aryabhata's method of solving linear equations and provides a number of illustrative astronomical examples. For many centuries, one of the approximations used for π was √10, which was criticized by Bhaskara I, who was concerned that an exact measure of the circumference of a circle in terms of diameter was not available and evidently believed that π was not rational.

Sridhar Acharya (also known as Sridhara) (870-930 CE), a Sanskrit pundit and philosopher, born in Bhurishresti (Bhurisristi or Bhurshut) village in Hooghly district of West Bengal (though some people believe that he was born in South India), was a highly esteemed mathematician who wrote several treatises on the two major fields of Indian mathematics, *pati-ganita* (mathematics of procedures, or algorithms) and *bija-ganita* (mathematics of seeds, or equations). Two of his treatises, for which he is famous, are *Trisatika* (also known as *Patiganitasara*) and *Patiganita*. His major work *Trisatika*, named so because it was written in three hundred slokas, discusses counting of numbers, measures, natural number, multiplication, division, zero, squares, cubes, fraction, rule of three, interest-calculation, joint business or partnership, and mensuration. He gave an exposition on zero. He wrote, "If 0 (zero) is added to any number, the sum is the same number; If 0 is subtracted from any number,the number remains unchanged; If 0 (zero) is multiplied by any number, the product is 0.)" He, however, said nothing about division of any number by 0. He separated algebra from arithmetic and wrote on practical applications of algebra. Sridhara was one of the first mathematicians to give a rule, known as *Sridhar Acharya's rule* or *Sridhar Acharya's formula*, to solve quadratic equations: If an equation can be represented as $ax^2 + bx + c = 0$, then the roots will be $x = [-b \pm (b^2 - 4ac)^{1/2}]/(2a)$.

Indian Mathematics

The original work is lost, but a quotation of *Sridhara's rule* from Bhaskara II reads as follows: "Multiply both sides of the equation by a known quantity equal to four times the coefficient of the square of the unknown; add to both sides a known quantity equal to the square of the coefficient of the unknown; then take the square root."

Aryabhata II (920-1000 CE), an Indian mathematician and astronomer, is best known for his work entitled *Mahasiddhanta* or *Aryasiddhanta* consisting of 18 chapters. The initial 12 chapters deal with matters related to mathematical astronomy and cover the topics that Indian mathematicians of that period had already worked on during this period. The remaining 6 chapters of the *Mahasiddhanta* form a separate part entitled *On the Sphere*, which discusses topics such as geometry, geography, and algebra with applications to the longitudes of the planets. In this work, Aryabhaṭa II also touches upon several arithmetical operations such as the four fundamental operations, operations with zero, extraction of square and cube roots, fractions, and the rule of three (which says that given three numbers a, b, and c, if one wants to find d such that a/b = c/d, then d = cb/a). But, his method of finding the cube root of a number was already given by Aryabhata I many years earlier. In *Mahasiddhanta*, Aryabhata II gives rules to solve the linear indeterminate equation of the type by = ax + c, which can be applied in a number of different cases such as when c is positive, when c is negative, when the number of the quotients of the mutual divisions is even, when this number of quotients is odd, etc. Aryabhata II prepared a sine table accurate up to five decimal places.

Bhaskara II (1114-1185 CE), who wrote extensively on all of the then known branches of mathematics, was one of the most accomplished of all India's great mathematicians. (Plofker, 2009, p. 182-207) He was the lineal successor of Brahmagupta as head of an astronomical observatory at Ujjain, the leading mathematical centre of ancient India. He filled many of the gaps in Brahmagupta's work, especially in obtaining a general solution to the so called Pell's equation ($x^2 - py^2 = 1$, where n is a given integer) and in giving many particular solutions. He derived a cyclic, *chakravala method* for solving indeterminate quadratic equations of the form $ax^2 + bx + c = y$ and $ax^2 + b = y^2$. (Goonatilake, 1998) He gave a proof of the Pythagorean theorem by calculating the same area in two different ways and then canceling out terms to get $a^2 + b^2 = c^2$. Bhaskara II anticipated the modern convention of signs (minus by minus makes plus, minus by plus makes minus). He is credited with explaining the previously misunderstood operation of division by zero. (Plofker, 2009, p. 197-198) He noticed that dividing one into two pieces yields a half, so $1 \div 1/2 = 2$. Similarly, $1 \div 1/3 = 3$. So, dividing 1 by smaller and smaller factions yields a larger and larger number of pieces. Ultimately, therefore, dividing one into pieces of zero size would yield infinitely many pieces, indicating that $1 \div 0 = \infty$ (the symbol for infinity). His work contains mathematical entities equivalent or approximately equivalent to infinitesimals, derivatives, the mean value theorem, and the derivative of the sine function. But, to what extent he anticipated the discovery of calculus is a controversial issue among historians of mathematics. To represent unknown quantities, Bhaskara II used letters much as in today's algebra, and solved indeterminate equations of 1st and 2nd degrees — solved quadratic equations reducing them to a single type. He examined regular polygons up to those having 384 sides, and obtained a satisfactory approximation of $\pi = 3.141666$. He is also famous for his book *Siddhanta Siromani* (1150 CE).

Indian Mathematics

Although almost all of his original work is lost, Madhava of Sangamagrama (1340-1425 CE), the founder of the Kerala School of Astronomy and Mathematics, is referred to in the work of later Kerala mathematicians as the source for several infinite series expansions, representing the first steps from the traditional finite processes of algebra to considerations of the infinite, with its implications for the future development of calculus and mathematical analysis. Madhava found the *Madhava–Leibniz series*, and through his application of this series, he obtained a value for π correct to as high as 13 decimal places. Madhava also found the *Madhava-Gregory series* to determine the arctangent, the *Madhava-Newton power series* to determine sine and cosine, and the Taylor approximation for sine and cosine functions. (Plofker, 2009, p. 217-253) Use of infinite series by Madhava for approximation of a range of trigonometric functions, which were further advanced by his Kerala School successors, laid a strong foundation for the later development of calculus and analysis. Either he or his students developed an early form of integration for simple functions; but, the Kerala School did not formulate a systematic theory of differentiation and integration. In the 16th century CE, Jyesthadeva consolidated many of the Kerala School's developments and theorems in his *Ganita-Yukti-Bhasa* (Rationales in Mathematical Astronomy). (Divakaran, 2007) Some historians suggest that Madhava's work, through the writings of the Kerala School, might have been transmitted to Europe via Christian missionaries and traders who were active around the port of Cochin then, and might have had influence on later European developments in calculus, but others say that there is no direct evidence of their results being transmitted outside Kerala. (Katz, 1995, p. 173-174)

2.4 Islamic Mathematics

Established across Persia, the Middle East, Central Asia, North Africa, Iberia, and in parts of India in the 8th century CE, the Islamic Empire drew on and fused together the mathematical developments of both Greece and India and made significant contributions towards mathematics. The House of Wisdom (Dar al-Ḥikma) was set up in Baghdad around 810 CE, and work started almost immediately on translating the major Greek and Indian works on mathematics and astronomy into Arabic. Most of the Islamic texts on mathematics were written in Arabic, but many of them were not written by Arabs, since much like the status of Greek in the Hellenistic world, Arabic was then used as the written language of non-Arab scholars throughout the Islamic world. Persians contributed to the world of Mathematics alongside Arabs.

Among the achievements of Muslim mathematicians during this period include the addition of the decimal point notation to the Arabic numerals, the development of algebra and algorithms, the discovery of all the modern trigonometric functions besides the sine, the invention of spherical trigonometry, introduction of *cryptanalysis* and *frequency analysis*, the development of analytic geometry and the earliest general formula for infinitesimal and integral calculus, the beginning of algebraic geometry, introduction of algebraic calculus, the first refutations of Euclidean geometry and the parallel postulate, the first attempt at a non-Euclidean geometry, the development of symbolic algebra, and numerous other advances in algebra, arithmetic, calculus, cryptography, geometry, number theory, and trigonometry. (O'Connor and Robertson, 2005)

Islamic Mathematics

The outstanding Persian mathematician Muḥammad ibn Musa al-Khwarizmi (780-850 CE), an early Director of the *House of Wisdom*, Baghdad, and one of the greatest of early mathematicians, wrote several important books on the Hindu-Arabic numerals and on methods for solving equations. His book *On the Calculation with Hindu Numerals* (Kitab al-Jamʿ wa-l-tafriq bi-ḥisab al-Hind), written about 825 CE, along with the four-volume book of al-Kindi on the number system, *On the Use of the Indian Numerals* (Ketab fi Isti'mal al-Adad al-Hindi), was instrumental in laying the foundation of a large part of modern arithmetic and spreading Indian mathematics and Indian numerals to the West. Al-Khwarizmi's most important contribution to mathematics was possibly his strong advocacy of the Hindu numerical system (1-9 and 0) and recognizing it as having the power and efficiency needed to revolutionize Islamic (and, later, Western) mathematics, which was soon adopted by the entire Islamic world, and later by Europe as well. Al-Khwarizmi converted (changed) Babylonian and Hindu numerals into a workable system that almost anyone could use. He gave the name to his mathematics as 'al-gabr' (meaning 'meeting' or 'relationship') which we know as 'algebra'. Al-Khwarizmi's book *On the Calculation with Hindu Numerals* was translated into Latin in the 12th century CE, as *Algoritmi de numero Indorum*, where 'Algoritmi', the translator's rendition of the author's name, gave rise to the word *algorithm* (Latin algorithmus) with a meaning 'calculation method'.

Al-Khwarizmi's other important contribution to mathematics was algebra. He introduced the fundamental algebraic methods of 'reduction' and 'balancing', referring to the transposition of subtracted terms to the other side of an equation, that is, the cancellation of like terms on opposite sides of the equation. This is the operation which al-Khwarizmi originally described as al-jabr (or al-gabr). (Boyer, 1991b, p. 229) He provided an exhaustive account of solving polynomial equations up to the second degree. (Boyer, 1991a, p. 230) He was the first to teach algebra in an elementary form and for its own sake. (Gandz, 1936) His algebra was also no longer concerned "with a series of problems to be resolved, but an exposition which starts with primitive terms in which the combinations must give all possible prototypes for equations, which henceforward explicitly constitute the true object of study." He also studied an equation for its own sake and "in a generic manner, in so far as it does not simply emerge in the course of solving a problem, but is specifically called on to define an infinite class of problems." (Rashed and Armstrong, 1994) This is how he helped create the powerful abstract mathematical language still used across the world today, and allowed a much more general way of analyzing problems other than just the specific problems previously considered by the Indians and Chinese. The word 'algebra' is derived from the title of one of his works, *Al-Kitab al-mukhtaṣar fi hisab al-jabr wa'l-muqabala* (The Compendious Book on Calculation by Completion and Balancing). 'Al-jabr' (also written as 'al-gabr') means 'completion', the process of removing negative terms from an equation, as for example, using al-Khwarizmi's own example, 'al-jabr' transforms $x^2 = 40x - 4x^2$ into $5x^2 = 40x$, and 'al-muqabala' means 'balancing', the process of reducing positive terms of the same power when they occur on both sides of an equation, as for example, two applications of 'muqabala' reduces $50 + 3x + x^2 = 29 + 10x$ to $21 + x^2 = 7x$. In the 12th century CE, the book was translated in Latin as *Liber algebrae et almucabala* by Robert of Chester (an English Arabist) hence 'algebra', and also by Gerard of Cremona (an Italian translator). A unique Arabic copy is kept at Oxford and was edited and translated by Frederic Rosen. (Rosen, 1831) A Latin translation is kept in Cambridge.

Islamic Mathematics

The Iraqi philosopher, mathematician, and musician Abu Yusuf Yaqub ibn Isḥaq aṣ-Ṣabbaḥ al-Kindi (801-873 CE) was a pioneer in *cryptanalysis* and *cryptology*. He is credited with developing a method whereby variations in the frequency of the occurrence of letters could be analyzed and exploited to break ciphers (i.e., *cryptanalysis by frequency analysis*). This was detailed in a text recently rediscovered in the Ottoman archives in Istanbul, *A Manuscript on Deciphering Cryptographic Messages*, which also covers methods of cryptanalysis, encipherments, cryptanalysis of certain encipherments, and statistical analysis of letters and letter combinations in Arabic. Al-Kindi's manuscript *On Deciphering Cryptographic Messages* contains the oldest known description of cryptanalysis by frequency analysis. Al-Kindi wrote many works on arithmetic which included manuscripts on Indian numbers, the harmony of numbers, lines and multiplication with numbers, relative quantities, measuring proportion and time, and numerical procedures and cancellation. Together with al-Khwarizmi's book *On the Calculation with Hindu Numerals*, al-Kindi's four-volume book *On the Use of the Indian Numerals* (Ketab fi Isti'mal al-Adad al-Hindi) (around 830 CE), is principally responsible for the diffusion of the Indian system of numeration in the Middle-East and the West. Among other works in geometry, al-Kindi wrote on the theory of parallels, and gave a lemma investigating the possibility of exhibiting pairs of lines in the plane which are simultaneously non-parallel and non-intersecting.

Abu Abdullah al-Battani (known as Albategnius in Latin) (858-929 CE), the Arab mathematician and astronomer, was the first to replace the use of Greek chords by sines and the first to develop the concept of cotangent and furnished their table in degrees. He introduced a number of trigonometric relations such as: $b \sin(A) = a \sin(90 - A)$ for right angled triangles with adjacent sides a and b, which is equivalent to $\tan A = a/b$; he used them for his astronomical studies. Al-Battani developed equations for calculating tangents ('shadows') and cotangents and compiled tables of them, introduced cotangent function for the first time, discovered the reciprocal functions of secant and cosecant, and produced the first table of cosecants, which he referred to as a "table of shadows" (in reference to the shadow of a gnomon), for each degree from 1° to 90°. (*Trigonometry*, Encyclopædia Britannica, retrieved 21-07-2008) His great book *Kitab az-Zij* (book of astronomical tables), which is now prevalent, was frequently quoted by many medieval astronomers, including Copernicus. The third chapter of the book deals with trigonometry. The book was several times translated into Latin and Spanish languages, including a Latin translation as *De Motu Stellarum* by Plato of Tivoli in 1116 CE, which was later reprinted with annotations by the German scholar Regiomontanus. (Chisholm, 1911)

The Persian mathematician Muhammad al-Karaji (953-1029 CE) worked to extend algebra still further in his treatise *Al-Fakhri fi'l-jabr wa'l-muqabala* (Book of al-Fakhri on the Art of Algebra) freeing it from its geometrical heritage, where he extends the methodology to incorporate integer powers and integer roots of unknown quantities. (Whitting, 1947) In this, al-Karaji introduced the monomials x, x^2, x^3, ... and $1/x$, $1/x^2$, $1/x^3$, ... and explained product rules among them. Moreover, he was the first to find the solutions of the equations $ax^{2n} + bx^n = c$. (Boyer, 1991a) Al-Karaji proved the sum formula for integral cubes by applying the method of proof by mathematical induction, and hence became the first to use the method of proof by induction to prove his results, (Katz, 1998) by proving that the first statement in an infinite sequence of statements is true, and then proving that, if any one statement in the sequence is true, then so is the next one.

Among other things, al-Karaji used mathematical induction to prove the *binomial theorem* around 1000 CE. A binomial is a simple type of algebraic expression that has just two terms which are operated on only by addition, subtraction, multiplication, and positive whole-number exponents, such as $(x+y)^2$. The coefficients needed when a binomial is expanded form a symmetrical triangle, usually referred to as *Pascal's Triangle* after the 17^{th}-century (CE) French mathematician Blaise Pascal, although many other mathematicians had studied it centuries before him in India, Persia, China, and Italy, including al-Karaji. Franz Woepcke, the German orientalist, mathematician, and the historian of mathematics, praised al-Karaji for being "the first who introduced the theory of algebraic calculus." (Woepcke, 1853)

About hundred years after al-Karaji, Omar Khayyam whose full name was Ghiyath al-Din Abu'l-Fath Umar ibn Ibrahim al-Nisaburi al-Khayyami (1048-1131 CE) (perhaps better known as a poet and the writer of the *Rubaiyat*, but an important mathematician and astronomer in his own right) generalized Indian methods for extracting square and cube roots to include fourth, fifth, and higher roots. Khayyam wrote a treatise, now lost, called *Problems of Arithmetic* involving the determination of n^{th} roots. (Youschkevitch and Rosenfeld 1973) In his *Treatise on Demonstrations of Problems of Algebra* (Maqalah fi al-jabra wa-al muqabalah) (1070 CE), Khayyam writes that methods for calculating square and cube roots come from India, and that he has extended them to the determination of roots of any order. Even more interestingly, he says that he has demonstrated the validity of his methods using proofs that "are purely arithmetic, founded on the arithmetic of the *Elements* (of Euclid)."

Khayyam also developed the triangular array of binomial coefficients (which later became known as *Pascal's Triangle* and stood at the base of probability) along with the *binomial theorem*, namely, the formula for the expansion of any binomial when raised to a power which is a positive integer, that is, the expansion of (x + y)n and determination of binomial coefficients, which is sometimes called *al-Khayyam's Formula*, $(x + y)^n = n!\sum x^k y^{n-k} / k!(n-k)!$. Binomial theorem was later generalized by Isaac Newton for all values of n, including fractions and negatives. Khayyam wrote on the triangular array of binomial coefficients (now known as *Pascal's triangle*) in his *Treatise on Demonstrations of Problems of Algebra* (1070 CE). (Coolidge, 1949)

Khayyam's *Algebra* is also known for its solution of the various cases of the cubic equation by finding the intersections of appropriately chosen conic sections (e.g., solved the cubic equation $x^3 + 200x = 20x^2 + 2000$ and found a positive root of this cubic by considering the intersection of a rectangular hyperbola and a circle). In this, he discussed the case "a cube, sides and numbers are equal to squares", or, in modern notation, $x^3 + cx + d = bx^2$. He carried out a systematic analysis of cubic problems, revealing that there were actually several different sorts of cubic equations. He was also the first to find the general geometric solution to cubic equations. Even though Khayyam succeed in solving cubic equations, and is usually credited with identifying the foundations of algebraic geometry, he was held back from further advances by his inability to separate the algebra from the geometry, and a purely algebraic method for the solution of cubic equations had to wait another 500 years and the Italian mathematicians Scipione del Ferro (1465-1526 CE) and Niccolo Fontana Tartaglia (1500-1557 CE).

In the late 11th century CE, Omar Khayyam wrote *Discussions of the Difficulties in Euclid*, a book about what he perceived as flaws in Euclid's *Elements*, especially the *parallel postulate* ("two convergent straight lines intersect and it is impossible for two convergent straight lines to diverge in the direction in which they converge") that opened the floodgates of non-Euclidean geometries. In trying to prove the parallel postulate, he accidentally proved properties of figures in non-Euclidean geometries. In this book, Khayyam also dealt with Euclid's definition of equality of ratios, extending Euclid's work to include the multiplication of ratios. Khayyam examined both Euclid's definition of equality of ratios and the definition of equality of ratios as proposed by earlier Islamic mathematicians such as al-Mahani which was based on continued fractions. He proved that the two definitions are equivalent. He also posed the question of whether a ratio can be regarded as a number but left the question unanswered. This work, on fractions and the multiplication of ratios, allowed the development of an entirely new aspect of mathematics. (Rashed and Vahabzadeh, 2000; O'Connor and Robertson, 1999; *Umar Khayyam*, Stanford Encyclopedia of Philosophy, 2011) Khayyam was also very influential in calendar reform.

Ibn al-Haytham (or Alhazen) (965-1040 CE), a renowned Arab physicist and mathematician, was the first to derive the formula for the sum of the fourth powers, using a method that is readily generalizable for determining the general formula for the sum of any integral powers. He performed integration in order to find the volume of a paraboloid, and was able to generalize his result for the integrals of polynomials up to the fourth degree. He thus came close to finding a general formula for the integrals of polynomials, but he was not concerned with any polynomials higher than the fourth degree. (Katz, 1995) Ibn al-Haitham was the first person to apply algebra to geometry, founding the branch of mathematics known as *analytic geometry*.

In number theory, ibn al-Haytham solved problems involving congruences using what is now known as *Wilson's theorem*, i.e., if p is prime, then $1 + (p - 1)!$ is divisible by p. In his *Opuscula*, ibn al-Haytham considers the solution of a system of congruences, and gives two general methods of solution. Another contribution by ibn al-Haytham to number theory was his work on perfect numbers. In his *Analysis and Synthesis*, he was possibly the first to state that every even perfect number is of the form $2^{n-1}(2^n - 1)$ where $2^n - 1$ is prime, but he was not able to prove this result successfully (Euler later proved it in the 18th century).

The Persian astronomer, scientist, and mathematician Nasir al-Din al-Tusi (1201-1274 CE) was perhaps the first to treat trigonometry as a separate mathematical discipline, distinct from astronomy. He made significant advances in spherical trigonometry. Building on earlier work by Greek mathematicians such as Menelaus of Alexandria (70-140 CE) and Indian work on the sine function, he gave the first extensive exposition of spherical trigonometry, including listing the six distinct cases of a right triangle in spherical trigonometry. One of his major mathematical contributions was the formulation of the famous law of sines for plane triangles, $a/(\sin A) = b/(\sin B) = c/(\sin C)$, although the sine law for spherical triangles had been discovered earlier by the 10th-century (CE) Persians Abul Wafa Buzjani and Abu Nasr Mansur.

In his *Al-risala al-shafiya'an al-shakk fi'l-khutut al-mutawaziya* (Discussion Which Removes Doubt about Parallel Lines), Nasir al-Din wrote detailed critiques of Euclid's *parallel postulate*. In this, he also wrote on Khayyam's attempted proof of the parallel *postulate* and tried to derive a proof of the postulate by contradiction. [Euclidean geometry is the geometry that satisfies all of Euclid's axioms, including the parallel postulate (that only one line may be drawn through a given point parallel to a given line); and a geometry where the parallel postulate does not hold is a non-Euclidean geometry.] He was also one of the first to consider the cases of elliptical and hyperbolic geometry. The first attempt at non-Euclidean geometry was made by Nasir al-Din's son Sdar al-Din al-Tusi (sometimes known as 'Pseudo Tusi') in a book which he wrote in 1298 CE based on Nasir al-Din's later thoughts, which presented one of the earliest arguments for a non-Euclidean hypothesis equivalent to the parallel postulate.

The Persian physicist and mathematician Kamal al-Din al-Farisi (1267-1319 CE) used theory of conic sections to solve optical problems, and made several important contributions to number theory such as on amicable numbers, factorization, and combinatorial methods. He noted the impossibility of giving an integer solution to the equation $x^4 + y^4 = z^4$ but he attempted no proof of this case of *Fermat's Last Theorem*, which states that $x^n + y^n = z^n$ has no non-zero integer solutions for x, y, and z when n > 2. Al-Farisi's most important work in number theory is on amicable numbers, which are a pair of numbers such that the sum of their proper divisors (not including itself) equals the other number. In modern notation, if S(n) denotes the sum of the aliquot parts of n, that is, the sum of its proper quotients, then the numbers m and n are called amicable provided that S(n) = m and S(m) = n. (Aliquot parts are integers which are exact divisors or factors of a number, for example, 3 is an aliquot part of 12.) In *Tadhkira al-ahbab fi bayan al-tahabb* (Memorandum for friends on the proof of amicability), al-Farisi gave a new proof of the following theorem by Thabit ibn Qurra on amicable numbers: Let $p_n = 3.2^n - 1$ and $q_n = 9.2^{2n-1} - 1$, for $n > 1$. If p_{n-1}, p_n, and q_n are prime numbers, then $a = 2^n p_{n-1} p_n$ and $b = 2^n q_n$ are amicable numbers. This was not a simple modification that al-Farisi made; rather he brought about a major change in approach to the whole area of number theory, through the introduction of new ideas concerning factorization and combinatorial methods.

Ghiyath al-Din Jamshid Masud al-Kashi (about 1380 - 1429 CE), the Persian astronomer and mathematician, showed a great versatility in numerical work, comparable to that later attained by late 16th-century Europeans. In fact, he devised a theory of numbers and techniques of computation that remained unmatched until recently. He is the first to use decimal point notation in arithmetic and Arabic numerals in the 15th century CE. (*Chronology of mathematics*, MacTutor History of Mathematics archive, University of St Andrews, Scotland) His book *Miftah al-Hussab* (The Key to Arithmetic) is notable for its inclusion of decimal fractions. He contributed to the development of decimal fractions not only for approximating algebraic numbers, but also for real numbers such as pi (π). He computed the value of π to the 16 decimal places of accuracy giving 2π as 6.2831853071795865 which was the best until about 1700 CE. His contribution to decimal fractions is so major that for many years he was considered as their inventor. Although not the first to do so, al-Kashi gave an algorithm for calculating n^{th} roots which is a special case of the methods given many centuries later by Ruffini and Horner. In fact, al-Kashi solved cubic equations by iteration and by trigonometric methods, and knew the method now known as Horner's method for the solution of general algebraic equations of higher order which generalizes the extraction of roots of higher order or ordinary numbers – a method which seems to point to Chinese influence. The binomial formula for a general positive integer exponent is found in his work. (Yadegari, 1980)

2.5 European Mathematics

During the period in which the Chinese, Indian, and Islamic mathematicians had been in the ascendancy, “Europe had fallen into the Dark Ages, in which science, mathematics, and almost all intellectual endeavour stagnated.” Intellectuals / scholars gave sole importance to studies in the humanities, such as philosophy and literature, and spent much of their energies arguing over subtle subjects in metaphysics and theology, such as "How many angels can stand on the point of a needle?"

2.5.1 Medieval European Mathematics

The Italian philosopher Boethius (480-524/525 CE) provided a place for mathematics in the curriculum (as a pedagogical necessity for the young) in the 6th century CE when he coined the term *quadrivium* to describe the study of an educational course introduced into monasteries consisting of a group of four topics / subjects: arithmetic, geometry, astronomy, and the theory of music. Up to the 12th centuries CE, European knowledge and study of the topics of the *quadrivium* was limited mainly to Boethius’ translations of some of the works of ancient Greek masters such as Nicomachus and Euclid. Boethius wrote *De institutione arithmetica*, a free translation from the Greek of Nicomachus's *Introduction to Arithmetic*; *De institutione musica*, also derived from Greek sources; translation of Ptolemy's *Almagest*; and a series of excerpts from Euclid's *Elements*. These translations contributed to medieval European education to a great extent. His works were theoretical, rather than practical, and were the basis of mathematical study until the recovery of Greek and Arabic mathematical works. In *De arithmetica*, Boethius began with modular arithmetic, such as even and odd, evenly-even, evenly-odd, and oddly-even; he then turned to unpredicted complexity by categorizing numbers and parts of numbers. In *De institutione musica*, he wrote on the relation of music to science, suggesting that the pitch of a note one hears is related to the frequency of sound.

All trades and calculations were done using the clumsy and inefficient Roman numeral system. But, by the 12th century CE, Europe, and particularly Italy, was beginning to trade with the East, and Eastern knowledge gradually began to spread to the West. Both Robert of Chester (an English Arabist) and Gerard of Cremona (an Italian translator of scientific books), who translated several historically important books from Arabic to Latin, translated al-Khwarizmi's famous book on algebra *Al-Kitab al-mukhtasar fi hisab al-jabr wa'l-muqabala* into Latin as *Liber algebrae et almucabala* (in 12th century CE), hence ‘algebra’; and the complete text of Euclid's *Elements* was translated in various versions by Adelard of Bath (1080-1152 CE), Herman of Carinthia (CE 1100-1160), and Gerard of Cremona. It was through translations of the Arabic texts that western Europe was able to develop its own mathematical traditions so rapidly, paving the way for the scientific revolution in the 17th century and thus to the scientific and technological world we take for granted today. Many of those Arabic texts were themselves translations of still earlier Greek works (from more than a thousand years earlier). (*The Mathematical Legacy of Islam*, Devlin's Angle, a monthly online column, for the Mathematical Association of America, July-August 2002)

In general, an increasing practical need for mathematics was created due to the great expansion of trade and commerce, and arithmetic entered much more into the lives of common people and was no longer confined to the academic sphere. The arrival of printing press in the mid-15th century CE also had a huge impact. Many books on arithmetic were published for the purpose of teaching computational methods to the business people for their commercial needs and thus mathematics gradually started acquiring a more important position in education.

European Mathematics

The Italian-born (and Algerian-trained) Leonardo of Pisa (1170-1250 CE), known by his nickname Fibonacci, was Europe's first great medieval mathematician. Leonardo is famous for the so-called *Fibonacci sequence* of numbers, which he introduced to Western European mathematics in his 1202-book *Liber Abaci* (although the sequence had been described earlier in Indian mathematics). However, his most important contribution to European mathematics was his role in spreading the use of the Hindu-Arabic numeral system all over Europe early in the 13th century CE, which soon replaced the Roman numeral system, and broke ground for great advances in European mathematics. In his book *Flos* (1225 CE), Fibonacci gave an accurate approximation to a root of $x^3 + 2x^2 + 10x = 20$, one of the problems that he was challenged to solve by Johannes of Palermo (a member of Emperor Frederick's court and a friend of Fibonacci's). This problem, however, was not made up by Johannes of Palermo, rather he took it from Omar Khayyam's algebra book where it is solved by means of the intersection of a circle and a hyperbola. In his number theory book *Liber quadratorum* (the book of squares), Fibonacci first notes that square numbers can be constructed as sums of odd numbers, essentially describing an inductive construction using the formula $n^2 + (2n+1) = (n+1)^2$. He also proved many interesting number theory results such as: there is no x, y such that $x^2 + y^2$ and $x^2 - y^2$ are both squares and $x^4 - y^4$ cannot be a square.

2.5.2 European renaissance mathematics

During the Renaissance period (14th-17th centuries), the development of mathematics and that of accounting were intertwined. While there is no direct relationship between algebra and accounting, the teaching of the subjects and the books published often intended for the children of merchants who were sent to reckoning schools [in Flanders (the northwestern part of present-day Belgium and adjacent parts of France and the Netherlands) and Germany] or abacus schools (known as abbaco in Italy), where they learned the skills useful for trade and commerce. There is probably no need for algebra in performing bookkeeping operations, but for complex bartering operations or the calculation of compound interest, a basic knowledge of arithmetic was mandatory and knowledge of algebra was very useful.

The Frenchman Nicole Oresme (also known as Nicolas Oresme) (1323-1382 CE), an important (but largely unknown and underrated) mathematician and scholar, used a system of rectangular coordinates centuries before his countryman René Descartes popularized the idea, as well as perhaps the first time-speed-distance graph. Also, leading from his research into musicology, Oresme was the first to use fractional exponents, and also worked on infinite series, being the first to prove the divergence of the harmonic series $1/1 + 1/2 + 1/3 + 1/4 + 1/5$ ... (i.e., not tending to a limit, other than infinity), something that was only replicated in later centuries by the Bernoulli brothers. He also worked on the notion of probability over infinite sequences, ideas which would not be further developed for the next three and five centuries, respectively.

The German scholar Regiomontatus was perhaps the most capable mathematician of the 15th century, his main contribution to mathematics being in the area of trigonometry. He helped separate trigonometry from astronomy, and it was largely through his efforts that trigonometry came to be considered an independent branch of mathematics. His book *De Triangulis*, in which he described much of the basic trigonometric knowledge which is now taught in high school and college, was the first great book on trigonometry to appear in print.

Mention should also be made of Nicholas of Cusa (or Nicolaus Cusanus), a 15th-century (CE) German philosopher, mathematician and astronomer, whose prescient ideas on the infinite and the infinitesimal directly influenced later mathematicians like Gottfried Leibniz and Georg Cantor. He also held some distinctly non-standard intuitive ideas about the universe and the Earth's position in it, and about the elliptical orbits of the planets and relative motion, which foreshadowed the later discoveries of Copernicus and Kepler.

The Italian mathematician Luca Pacioli's *Summa de Arithmetica, Geometria, Proportioni et Proportionalità* (Review of Arithmetic, Geometry, Ratio and Proportion) was first printed and published in Venice in 1494. It included a 27-page treatise on bookkeeping, "Particularis de Computis et Scripturis" (Italian: "Details of Calculation and Recording"). It was written primarily for, and sold mainly to, merchants who used the book as a reference text, as a source of pleasure from the mathematical puzzles it contained, and to aid the education of their sons. In *Summa Arithmetica*, Pacioli introduced symbols for plus and minus for the first time in a printed book, symbols that became standard notation in Italian Renaissance mathematics. *Summa Arithmetica* was also the first known book printed in Italy to contain algebra. It is important to note that Pacioli himself had borrowed much of the work of the Italian mathematician Piero Della Francesca (1412-1492 CE) whom he plagiarized.

In Italy, during the first half of the 16th century, Scipione del Ferro and Niccolò Fontana Tartaglia discovered solutions for cubic equations. Gerolamo Cardano published them in his 1545-book *Ars Magna*, together with a solution for the quartic equations, discovered by his student Lodovico Ferrari. In 1572 CE, Rafael Bombelli published his *L'Algebra* in which he showed how to deal with the imaginary quantities that could appear in Italian Renaissance mathematician Cardano's formula for solving cubic equations.

The Dutch mathematician Simon Stevin's book *De Thiende* ('the art of tenths') (1585 CE) contained the first systematic treatment of decimal notation, which influenced all later works on the real number system.

Driven by the demands of navigation and the growing need for accurate maps of large areas, trigonometry grew to be a major branch of mathematics. The German mathematician Bartholomaeus Pitiscus was the first to use the word, publishing his Trigonometria in 1595. Regiomontanus's table of sines and cosines was published in 1533. (Grattan-Guinness, 1997)

European Mathematics

If Gauss is the Prince, Leonhard Euler (1707-1783 CE) is the King. A leading Swiss mathematician, Euler is regarded as the greatest mathematician to have ever walked this planet. It is said that all mathematical formulas are named after the next person after Euler to discover them. In his day he was ground breaking and on par with Einstein in genius. Among his diverse works, the most notable was the introduction of mathematical notations. He introduced the concept of a function and how it is written by writing f(x) to signify the function 'f' applied to the argument 'x'. He also pioneered the modern notation for the trigonometric functions, the letter 'e', for the base of the natural logarithm (known as Euler's number), the Greek letter 'Σ' (sigma) for summations, and the letter 'i' to signify the imaginary unit, as well as the symbol π (pi) for the ratio of a circles circumference to its diameter. All of which play a huge bearing on modern mathematics, from the everyday to the incredibly complex.

In analysis, Euler initiated the use of the exponential functions and logarithms in analytic proofs, discovered ways to state various logarithmic functions using power series, and effectively defined logarithms for negative and complex numbers. Through these achievements, he widened the scope of mathematical applications of logarithms to a great extent. By inventing the gamma function, Euler explained in detail the theory of higher transcendental functions. He introduced a novel approach for solving quartic equations (a fourth-order polynomial equation of the form $ax^4 + bx^3 + cx^2 + dx + e = 0$). His discovery of a technique to calculate integrals with complex limits helped development of modern complex analysis. He also discovered the calculus of variations along with the Euler–Lagrange equation. In number theory, Euler proved Fermat's little theorem, Newton's identities, and Fermat's theorem on sums of two squares. He also individually contributed to Lagrange's four-square theorem and made significant value addition to the theory of perfect numbers, which had always been a captivating topic for several mathematicians.

During the Renaissance, the desire of artists to represent the natural world realistically together with the rediscovered philosophy of the Greeks led artists to study mathematics. They were also the engineers and architects of that time, and so had need of mathematics in any case. The art of painting in perspective, and the developments in geometry that involved, were studied intensely. (Kline, 1953)

2.6 Mathematics in Modern Era

The modern period of mathematics dates from the beginning of the 19th century, and its dominant figure is of course Carl Friedrich Gauss (1777-1855 CE), the prince of mathematics. Throughout the 19th century, mathematics became increasingly abstract. Leaving aside his many contributions to science, in pure mathematics he did revolutionary work on functions of complex variables, in geometry, and on the convergence of series. At the end of his college years, Gauss made an amazing discovery when he found that a regular polygon with 17 sides could be drawn using just a compass and straight edge, that, up to this time, mathematicians had believed was impossible. This made him so happy about and proud of his discovery that he gave up his intention to study languages and turned to mathematics.

In the area of geometry, Gauss made fundamental contributions to differential geometry, did much to found what was first called 'analysis situs' but is now called topology, and anticipated (although he did not publish his results) the great breakthrough of non-Euclidean geometry. Some of Gauss' ideas were a hundred years ahead of their time, and touched on many different parts of the mathematical world, including geometry, number theory, calculus, algebra, Gauss elimination, the method of least squares, probability theory, the Gaussian distribution of statistics, etc. He gave the first satisfactory proofs of what is known as the *fundamental theorem of algebra*, that is, every algebraic equation has at least one root or solution, which had challenged mathematicians for centuries. He proved the quadratic reciprocity law, a theorem about modular arithmetic which gives conditions for the solvability of quadratic equations modulo prime numbers, in a number of ways. He is widely regarded as one of the greatest mathematicians of all time.

Gauss founded the modern theory of numbers, gave the first clear explanation / interpretation of complex numbers, and studied the functions of complex variables. The concept of number was further extended by the Irish mathematician W. R. Hamilton (1805-1865) whose theory of quaternions (a number system that extends the complex numbers) provided the first example of a non-commutative algebra (i.e., one in which $ab \neq ba$). (His work proved significant for the development of quantum mechanics.) Hamilton's work was generalized by H. G. Grassmann (1809-1877), a German polymath who showed that several different consistent algebras may be derived by choosing different sets of axioms governing the operations on the elements of the algebra.

Remark: *Quaternions have had a revival since the late 20th century because of their utility in describing spatial rotations — the representations of rotations by quaternions are more compact and quicker to compute than the representations by matrices — and due to such reasons Quaternions are now used in computer graphics, computer vision, robotics, control theory, signal processing, attitude control, physics, bioinformatics, molecular dynamics, computer simulations, and orbital mechanics. For example, it is common for the attitude-control systems of spacecraft to be commanded in terms of quaternions. Quaternions have received another boost from number theory because of their relationships with the quadratic forms.*

The eminent German mathematician Bernhard Riemann (1826-1866 CE) worked on a different kind of non-Euclidean geometry called *elliptic geometry*, as well as on a generalized theory of all the different types of geometry. Riemann's innovative published works constructed the base for what is known as modern mathematics and research areas including analysis and geometry. These works finally proved to be very useful in the theories of algebraic geometry, Riemannian geometry, and complex manifold theory. Breaking away completely from all the limitations of 2 and 3 dimensional geometries, whether flat or curved, Riemann began to think in higher dimensions. His exploration of the zeta function in multi-dimensional complex numbers revealed an unexpected link with the distribution of prime numbers, and his famous Riemann Hypothesis, still unproven after 150 years, remains one of the world's great unsolved mathematical mysteries and the testing ground for new generations of mathematicians.

Remark: The Riemann zeta function or Euler-Riemann zeta function is a function of a complex variable s given by the infinite series $\zeta(s) = 1 + 2^{-s} + 3^{-s} + 4^{-s} + \cdots$. *When s = 1, this series is called the harmonic series and its sum is infinite; for values of s larger than 1, the series converges to a finite number as the successive terms are added; if s is less than 1, the sum is again infinite. Euler introduced and studied this function, as a function of a real argument, in 1737, without using complex analysis, which was not available then. In 1859, Riemann extended the Euler definition to a complex variable, proved its meromorphic continuation and functional equation, and established a relation between its zeros and the distribution of prime numbers. The Riemann zeta function plays a pivotal role in analytic number theory and has applications in physics, probability theory, and applied statistics.*

Charles Babbage (1791-1871 CE), the English mathematician and inventor, designed a complex machine called the Analytical Engine that could automatically perform computations based on a program of instructions stored on punched cards or tape. His large 'difference engine' of 1823 was able to calculate logarithms and trigonometric functions, and was the true forerunner of the modern electronic computer. The Engine was continuously redesigned and developed from 1833 until his death.

His large 'difference engine' of 1823 was able to calculate logarithms and trigonometric functions, and was the true forerunner of the modern electronic computer. The Engine was continuously redesigned and developed from 1833 until his death.

The British mathematician and logician George Boole (1815-1864 CE) devised an algebra (now called *Boolean algebra* or *Boolean logic*), in which the only operators are AND, OR, and NOT, and which could be applied to the solution of logical problems and mathematical functions. He also described a kind of binary system which used just two objects, 'on' and 'off' (or 'true' and 'false', 0 and 1, etc), in which 1 + 1 = 0, and carry 1 to the next more significant bit (short for binary digit). Boolean algebra was the starting point of modern mathematical logic and ultimately led to the development of computer science and this logical theory acts as the basis of modern digital computer and other electronic devices.

The German mathematician Georg Ferdinand Ludwig Philipp Cantor's (1845-1918 CE) achievement in mathematics was outstanding. He revolutionized the foundation of mathematics with set theory, which enabled the rigorous treatment of the notion of infinity, and which has since become the common language of nearly all mathematics. Before Cantor it was generally felt that infinity as an actuality did not make sense; one could only speak of a variable increasing without bound as that variable going to infinity. That is to say, it was felt that $n \to \infty$ makes sense but $n = \infty$ does not. Cantor not only found a way to make sense out an actual, as opposed to a potential, infinity but showed that there are different orders of infinity, which was a shock to people's intuition. Georg Cantor is also credited with defining the *cardinal numbers* and the *ordinal numbers* and their arithmetic. He is also responsible for advancing the study of the trigonometric series.

Cantor's work on set theory was extended by another German mathematician Richard Dedekind (1831- 1916 CE) who made enormous contribution in the field of abstract algebra (especially, algebraic number theory, theory of real numbers, and ring theory). He developed a major redefinition of irrational numbers in terms of arithmetic concepts, and defined concepts such as *similar sets* and *infinite sets*. Dedekind also came up with the notion called a *Dedekind cut*, his most celebrated work, which is now a standard definition of the real numbers and based on which he developed a rigorous method for constructing the set of real numbers. He showed that any irrational number divides the rational numbers into two classes or sets, the upper class being strictly greater than all the members of the other lower class. His work also heavily influenced the field of algebraic number theory, where one of the key objects of study is a particular type of ring called a *Dedekind domain*. In algebra, he coined the term *ideal* for one of the fundamental notions in ring theory.

The French mathematician Joseph Fourier's (1768-1830 CE) study of infinite sums in which the terms are trigonometric functions was another important advance in mathematical analysis. Periodic functions that can be expressed as the sum of an infinite series of sines and cosines are known today as *Fourier Series*, and they are still powerful tools in pure and applied mathematics. Following Leibniz, Euler, Lagrange, and others, Fourier also contributed towards defining exactly what is meant by a function, although the definition that is found in texts today — defining it in terms of a correspondence between elements of the domain and the range — is usually attributed to the 19th -century German mathematician Peter Dirichlet. Besides the above, Fourier left an unfinished work on determinate equations, completed and published by Claude-Louis Navier in 1831 CE, which included a demonstration of Fourier's theorem on the position of the roots of an algebraic equation. The other significant physical contribution was Fourier's proposal of his partial differential equation for conductive diffusion of heat, which is taught to every student of mathematical physics till date.

Along with Riemann, the French mathematician Augustin-Louis Cauchy (1789-1857 CE) and the German mathematician Karl Weierstrass (1815-1897 CE) completely redeveloped calculus in an even more rigorous fashion, leading to the development of mathematical analysis, a branch of pure mathematics largely concerned with the notion of limits (whether it be the limit of a sequence or the limit of a function) and with the theories of differentiation, integration, infinite series, and analytic functions. In 1845, Cauchy also proved a fundamental theorem of group theory named after him as *Cauchy's theorem*, which he discovered while examining permutation groups. His theories on functions of complex variables have played substantial role in subjects ranging from applied mathematics to aeronautics. His significant papers on error theory act as valuable asset for the domain of science. Cauchy was the first mathematician who developed definitions and rules for mathematics. He introduced the definitions of the integral and rules for series convergence. Carl Jacobi (1804-1851 CE), the German mathematician, also made important contributions to analysis, determinants, and matrices, and especially to theory of periodic functions and elliptic functions and their relation to the elliptic theta function.

The breakthrough of non-Euclidean geometry, anticipated by C. F. Gauss, was made by the Russian mathematician Nikolai Ivanovich Lobachevsky (1792-1856 CE) and independently by **the** Hungarian mathematician Janos Bolyai (1802-1860 CE), the son of a close friend of Gauss, whom each proceeded by establishing the independence of Euclid's fifth (parallel) postulate and showing that a different, self-consistent geometry could be derived by substituting another postulate in its place. Yet another non-Euclidean geometry was invented by Bernhard Riemann in 1854, whose work also laid the foundations for the modern tensor calculus description of space, so important in the general theory of relativity. In the year 1905 Albert Einstein determined that the laws of physics were the same for all non-accelerating observers, and that the speed of light in a vacuum was independent of the motion of all observers. This was the theory of special relativity. This theory introduced a new framework for all of physics and proposed new concepts of space and time. The General Theory of Relativity is, as the name indicates, a generalization of the Special Theory of Relativity. General relativity is a theory of gravitation that was developed by Albert Einstein between 1907 and 1915. According to the theory of general relativity, the observed gravitational effect between masses results from the warping of space-time.

Such developments continued with the group theory of M. S. Lie (1842-1899 CE) and reached full expression in the wide scope of modern abstract algebra. Lie made major advances in the theory of continuous groups of transformations and differential equations. Lie groups and Lie algebras are named after him. Number theory further developed in the latter half of the 19th century through significant contributions of Cantor, Weierstrass, and the German mathematician W. R. Dedekind (1831-1916 CE).

Yet another influence of Gauss was his insistence on rigorous proof in all fields of mathematics. In analysis, his close examination of the foundations of the calculus resulted in *Cauchy's theory of limits* (1821 CE), which in turn yielded new and clearer definitions of continuity, the derivative, and the definite integral. Raising new questions about these concepts and showing that ultimately the foundations of analysis rest on the properties of the real number system, Weierstrass took another important step toward rigor.

The 20th century saw continuation of the trend of the 19th century toward increasing generalization and abstraction, with the elements and operations of systems being defined so broadly that their interpretations connect such areas as algebra, geometry, and topology. The use of formal axiomatics has been the key to this approach, where an emphasis on such logical concepts as consistency and completeness has been accepted repealing the notion of axioms as 'self-evident truths'. The roots of formal axiomatics lie in the discoveries of alternative systems of geometry and algebra in the 19th century; the approach was first systematically undertaken by a brilliant German mathematician David Hilbert (1862-1943 CE) in his work on the foundations of geometry (1899). Hilbert discovered and developed a broad range of fundamental ideas in many areas including invariant theory and the axiomatization of geometry. He was responsible for several theorems and some entirely new mathematical concepts, as well as overseeing the development of what amounted to a whole new style of abstract mathematical thinking. He also formulated the theory of *Hilbert spaces*, one of the foundations of functional analysis.

Mathematics in Modern Era

The stress on deductive logic inherent in this view of mathematics and the discovery of the interconnections between the various branches of mathematics and their ultimate basis in number theory led to intense activity in the field of mathematical logic after the turn of the century. Rival schools of thought grew up under the leadership of Hilbert, Bertrand Russell (the British philosopher and mathematician, 1872-1970 CE), A. N. Whitehead (the English mathematician and philosopher, 1861-1947 CE), and L. E. J. Brouwer (the Dutch mathematician and philosopher, 1881-1966 CE). Important contributions in the investigation of the logical foundations of mathematics were made by Kurt Gödel (the Austrian-American mathematician, and philosopher, 1906-1978 CE) and Alonzo Church (the American mathematician and logician, 1903-1995 CE).

The British mathematician Godfrey Harold Hardy (1877-1947 CE), known for his achievements in number theory and mathematical analysis, and his young Indian student Srinivasa Ramanujan, were just two of the great mathematicians of the early 20th century who earnestly devoted themselves to solving problems of the previous century, such as the *Riemann hypothesis.* Although they came close to reaching a solution, they too were defeated by those most intractable of problems. However, Hardy is credited with reforming British mathematics (sunk to a low ebb at that time) by bringing rigour into it, which was previously a characteristic of French, Swiss, and German mathematics. British mathematicians had remained largely in the tradition of applied mathematics, in hostage to the reputation of Isaac Newton. Sir Isaac Newton (1642-1727 CE), an English physicist and mathematician, and the culminating figure of the 17th-century scientific revolution, is one of the most influential scientists of all time. Newton was the original discoverer of the infinitesimal calculus. His *Philosophiae Naturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy) (1687 CE) was one of the most important and influential works in physics of all times. Hardy aggressively promoted his conception of pure mathematics, in particular against the hydrodynamics which was an important part of Cambridge mathematics.

Hardy's collaboration with J. E. Littlewood in mathematical analysis and analytic number theory led to quantitative progress on the *Waring's problem*, as part of the *Hardy-Littlewood circle method*, one of the most basic methods in additive number theory. (Waring's problem is a problem in the theory of numbers formulated (without proof) by the English mathematician E. Waring in 1770 CE which asks whether each natural number *k* has an associated positive integer *s* such that every natural number is the sum of at most s k^{th} powers of natural numbers.) Their proof of some results and some notable conditional results in prime number theory was a major factor in the development of number theory as a system of conjectures, for example, the first and second *Hardy-Littlewood conjectures*. Hardy-Littlewood collaboration is among the most successful and famous collaborations in the history of mathematics. The Danish mathematician Harald Bohr said, "Nowadays, there are only three really great English mathematicians: Hardy, Littlewood, and Hardy–Littlewood." Hardy is also known for formulating the *Hardy-Weinberg principle*, a basic principle of population genetics, independently from Wilhelm Weinberg in 1908.

His geneticist companion Reginald Punnett with whom he used to play cricket introduced the problem to him, and Hardy thus became the somewhat unwitting founder of a branch of applied mathematics. His famous work on integer partitions with his collaborator Srinivasa Ramanujan, known as the *Hardy-Ramanujan asymptotic formula*, “has been widely applied in physics to find quantum partition functions of atomic nuclei (first used by Niels Bohr) and to derive thermodynamic functions of non-interacting Bose-Einstein systems. Though Hardy wanted his mathematics to be ‘pure’ and free from any application, much of his work has found applications in other branches of science.” Possibly because of his detestation of war and the military uses to which mathematics had been applied, he wished his work to be recognized as *pure mathematics*. He made a number of statements similar to that in his 1940-essay *A Mathematician's Apology*: *"I have never done anything 'useful'. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world."*

Srinivasa Ramanujan (1887-1920 CE) proved himself to be one of the most brilliant (if somewhat undisciplined and unstable) minds of the century. Ramanujan is undoubtedly the most celebrated Indian Mathematical genius, whose chief contribution in mathematics lies mainly in analysis, game theory, and infinite series. His contributions to the theory of numbers include pioneering discoveries of the properties of the partition function. Ramanujan showed that any big number can be written as sum of not more than four prime numbers. He showed how to divide the number into two or more squares or cubes. He investigated the series $\sum(1/n)$ and calculated Euler's constant to 15 decimal places. He began to study the Bernoulli numbers, although this was entirely his own independent discovery. During an illness in England, his mentor G. H. Hardy visited Ramanujan in the hospital. When Hardy remarked that he had taken taxi bearing number 1729, a singularly unexceptional number, Ramanujan immediately responded that this number was actually quite remarkable. It is the smallest integer that can be written in the form of sum of cubes of two numbers in two ways, i.e., $1729 = 9^3 + 10^3 = 1^3 + 12^3$; since then the number 1729 is called *Ramanujan’s number*. Such numbers are also called the *taxicab numbers*. The n^{th} taxicab number Ta(n) is the smallest number representable in n ways as a sum of positive cubes. For example, the number $1729 = Ta(2) = 1^3+12^3 = 9^3 + 10^3$.

About the value of zero divided by zero, Ramanujan thought that it can be anything since the zero of the denominator may be several times the zero of the numerator and vice versa and that the value cannot be determined. For the calculation of the value of π, he identified several efficient and rapidly converging infinite series some of which could compute 8 additional decimal places of π with each term in the series. Ramanujan also carried out major investigations in the areas of gamma functions, modular forms, divergent series, hypergeometric series, and prime number theory. Some of his original and highly unconventional results, such as the Ramanujan prime and the Ramanujan theta function, have inspired vast amounts of further research and have found applications in fields as diverse as crystallography and string theory.

3. Modernism and Post-Modernism in Mathematics

Starting in Italy, the European Renaissance (about 1420 to 1630 CE) was an important transition period beginning between the Late Middle Ages and Early Modern Times. The modern era began roughly in the 16th century CE and continued up to the term 'modern' was coined shortly before 1585 to describe the beginning of a new era. According to Jeremy Gray, the English mathematician and historian of mathematics, "*Modernism* is defined as an autonomous body of ideas, having little or no outward reference, placing considerable emphasis on formal aspects of the work and maintaining a complicated — indeed, anxious — rather than a naive relationship with the day-to-day world." (Gray, 2008) The Major components of modernism are *rationalism* (faith in knowledge through reason), *empiricism* (faith in knowledge through scientific method), and *materialism* (faith in a purely physical universe).

In his *Autobiography*, Bertrand Russell (CE 1872-1970), the famed British philosopher, logician, mathematician, historian, social critic, political activist, and Nobel Laureate, wrote: "Mathematics, rightly viewed, possesses not only truth, but supreme beauty-a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of stern perfection, such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry. What is best in mathematics deserves not merely to be learned as a task, but to be assimilated as a part of daily thought, and brought again and again before the mind with ever-renewed encouragement. Real life is, to most men, a long second-best, a perpetual compromise between the real and the possible; but the world of pure reason knows no compromise, no practical limitations, no barrier to the creative activity embodying in splendid edifices the passionate aspiration after the perfect from which all great work springs. Remote from human passions, remote even from the pitiful facts of nature the generations have gradually created an ordered cosmos, where pure thought can dwell as in its natural home, and where one, at least, of our nobler impulses can escape from dreary exile in the natural world" (Russell, 1907, 1967).

The so-called absolutist schools of philosophy of mathematics believe that mathematics is an absolute, certain, universal, and independent of human knowledge. For instance, Euclid's *Elements* was taken as an absolute truth for over 2000 years. The traditional philosophies of mathematics, that is, (i) l*ogicism* (the view that mathematics is an extension of logic), *(ii) formalism* [the concept that any branch of mathematics can be axiomatized in a formal language (by a set of meaningless symbols and arbitrarily pre-assigned rules) as a complete consistent formal system, refuting the role of experience], *(iii) intuitionism* (a view in which to be true is to be experienced, i.e., the view that the subject matter of mathematics consists of the mental or symbolic constructions of mathematicians rather than independent and timeless abstractions, as is held in Platonism), and *(iv) Platonism* or *mathematical Platonism* (the metaphysical view that mathematics is static and based on some transcendental abstract entities which are perfect, eternal, unchangeable, are out of space and independent of humanity). They have tried to establish the certainty of mathematical truth. (Moslehian, 2005)

In the sense the term is used by working mathematicians these days, Modern mathematics took shape primarily in Germany and France in the period from 1890 to 1930 CE, when remarkably new concepts were introduced, new methods were employed, and whole new areas of specialization emerged, while other themes were dismissed. Meanwhile, the nature of mathematical truth and even the consistency of mathematics were challenged, as mathematicians, logicians, and philosophers confronted the subject's very foundations. Incidentally, the same period witnessed radical societal, cultural, technological, and scientific changes, almost across the board. In ***Plato's Ghost,*** **the** English mathematician Jeremy Gray (b. 1947) contends that the best way to explain this remarkable transformation of mathematics is in terms of certain overarching 'modernist' features or themes. Because the term **modernism** is usually reserved for the changes that took place in literature and the arts during the period in question and because there is no apparent connection between the content of the new mathematics and that of the representative artistic creations, question naturally arises about what ties them together. Gray's explanation (in his Plato's Ghost) is that as quoted above in the first paragraph of this section.

On the contrary, schools of philosophy of mathematics for conceptual change claim that mathematics is fallible, corrigible, and a changing social product. Mathematical truths are never absolute and certain and they should be understood as relative to a background pre-assumed system, e.g., 2+2 = 4 is not an absolute truth, since in the system base-3 modular arithmetic, we have 2+2 =1 and a carry 1. (Ernest, 1998; Moslehian, 2005)

But the academic movement known as *postmodernism*, which is present now in all disciplines, takes a different view. As the American historian Gertrude Himmelfarb (b. 1922) alerts us, "The animating spirit of postmodernism is a radical skepticism and relativism that rejects any idea of truth, knowledge, reason, or objectivity. More important, it refuses even to aspire to such ideas, on the ground that they are not only unattainable but undesirable — that they are, by their very nature, authoritarian and oppressive." (Himmelfarb, 1995) Albert Einstein (1879-1955 CE) stated that "as far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."

A mathematician is quite likely to be motivated by the Platonic view that mathematics is external to the human mind, that mathematical truth is discovered, and that it has the desirable quality of being absolute (within a given system of axiomatic assumptions). This traditional view is today being deconstructed by some mathematicians and by many mathematic educators. (Ernest, 1991) The notion of mathematics as objective and eternal is today being replaced, among mathematics educators, by the postmodernist notion of 'social constructivism'.

According to 'social constructivism', knowledge is subjective, not objective; rather than being found by careful investigation of an actually existing external world, it is 'constructed' (i.e., created) by each individual, according to his unique needs and social setting. Absolutism is deliberately replaced by cultural relativism, as if 2 + 2 = 5 were correct as long as one's personal situation or perspective required it to be correct.

Postmodernism was coined to refer to the major changes in the 1950s and 1960s (i.e., late 20th century onwards) in economy, society, culture, and philosophy.Postmodernism has such features as the tolerance of ambiguity and disorder, stressing on skepticism and nihilism, the mixing of styles and manners, rejection of ultimate reality and absolute truth, lack of determinism and dogmatism. It emphasizes negative critical capacity and looks for such oppositions as good-bad, truth-fiction, and science-myth. It is a refusal of any base for moral and of any already given meaning in the universe and a reaction against any naive confidence in progress.

Postmodernism resists the knowledge based on rationality, objectivity, and the thought (based on what is true, good or beautiful). In postmodernism, knowledge has an essentially pluralistic character in the sense that diversity, divergence, contradictory, and incommensurable interpretations contest each other without cancelling each other out. In postmodernism, intellect, morality, and reason are replaced by will, relativism, and emotion, respectively. Like many other subjects, e.g., art, architecture, music, film, literature, sociology, communications, fashion, science, technology, philosophy, mathematics has also been influenced by postmodernism in recent era.

The most famous school in this direction is *Humanism*, a naturalistic philosophy inspired by art and founded on human experience. Humanism is a recent view claiming that mathematics is a social-cultural-historical product based on what we can actually observe; it is fallible (capable of making mistakes) being done by fallible people and quasi-empirical analogous to natural sciences. Mathematical truths are uncertain like any other truths. (Davis and Hersh, 1981) Mathematics is like law, like money, like religion, and like all those other things which are very real, but only as a part of collective human consciousness, so there is no mathematics without human beings.

Abstract ideas do not fall from heaven; they are considered as a human endeavour. Mathematical knowledge cannot be given a final and fully rigourous form; in fact, what separates formal from informal mathematics is only their degree of rigour and formality. Mathematics is not infallibility, since mathematics makes mistakes; is not unique, because there are different approaches to investigate the same thing; is not certain because of a lack of rigour practiced by many mathematicians. Mathematics is constructed, not discovered; and is contextual not foundational. Humanist philosophy is educationally beneficial, since it is about and a part of our life; it is alive, growing, and accessible; and everyone could learn and like it. (Davis and China, 1985; Davis and Hersh, 1981) Besides, applications of mathematical knowledge to empirical sciences can be fruitful in humanism. Moreover, as noted by Lakatos, mathematics is developed through the application of methodologies within rational research programs. (Lakatos, 1976)

In the natural sciences, mathematics is the absolute ruler; if one breaks the mathematical rules, he / she is no longer doing astronomy, or physics, or whatever. But in the human sciences, mathematics is just a tool — a useful one to be sure, and getting more useful all the time, but after all, just a tool. In his book *Goodbye, Descartes: The End of Logic and the Search for a New Cosmology of the Mind*, Keith Devlin attempts to articulate what he saw as the inherent limitations of the mathematical approach (the 'Cartesian approach') in the human sciences. (Devlin, 1998)

Mathematicians need more interdisciplinary communication to appreciate about how important it is to understand that there are many different kinds of truths, many different ways to make sense of our experience in the world. They should reassess the role that mathematics can play in the human sciences. (Devlin, 2011) "Although mathematical proof is being sought precisely because of the certainty it is ordinarily held to grant, constructing proofs of computer systems correctness again turns out to be no simple 'application' of mathematics. It involves negotiating what proof consists of." (MacKenzie, 1993)

Postmodern mathematics, to which humanism is compatible, has turned from abstraction to representation and from control to indeterminacy. Moreover certainty has become an unattainable idea. It is changing its style and its role in our culture. The mathematical objects are invented from the needs of science and life by humans, and have no sense beyond their cultural meanings. In 1930, Kurt Friedrich Gödel, the well-known Austrian-American mathematician and philosopher-scientist (1906-1978) proved that for any large enough axiomatic system to prove consistency we must add further assumptions to the set of our axioms. In addition, as a Hungarian philosopher of mathematics and science Imre Lakatos showed, a mathematical proof depends on a set of assumptions which are assumed by humankind, not absolute certainty. (Lakatos, 1978) Hence any attempt to establish the certainty of mathematical knowledge fails.

Computers are changing the way mathematicians discover and prove ideas. Investigators have proposed a computational proof that offers only the probability, not the certainty, of truth. Skillfully performed and analyzed computational experiments can yield more results than the outdated conjecture-proof method. (Zawadowski, 1990)

Although scientists claim to be guided by rationality, the rules of logic are nothing but socially prescribed ways of thinking. Postmodernism emphasizes fuzzy logic as an approach to decision based on 'degrees of truth' rather than the usual 'true-false'. Lutfy Zadeh's *Fuzzy set theory* resembles human reasoning in its use of approximate information and partial truth. Hence it is ideal for controlling nonlinear systems and for modeling complex systems where ambiguity and uncertainty are common.

From a postmodern perspective, knowledge, and in particular mathematics, is dynamic and one should always rethink and deconstruct his / her beliefs and tools, so that there is an emphasis on criticism rather than evaluation. Knowledge is free from any dependence on the concept of objective truth; instead, one's beliefs should constantly be expressed lightly and seen as temporary theories.

Knowledge is characterized by its utility and is functional, i.e., things are learnt, not to know, but to use them. Today mathematics has penetrated the so-called fields of social sciences, not to mention the natural sciences, and contributes to them – postmodern mathematics is taking over everything one has, does, or wishes. For example, while shopping at a supermarket, the barcode printed on the product is scanned and the amount is charged. The barcode not only calculates the price, it also sends information on the purchased product and time of purchase etc. to the supermarket headquarters instantaneously. Furthermore, when a point card is produced, the purchaser's gender, age, etc. and information indispensible for marketing strategies such as product development and product delivery times, are known. As huge amounts of such data comes up on a daily basis, how to effectively extract and read beneficial information and patterns from large volumes and large-scale data is the key to setting up marketing strategies. Various data analysis methods developed in statistical science and machine learning are used for this purpose.

Remark: *Social mathematics is a way of presenting numbers in a real-life — familiar context that helps people see the story behind them. It is a tool that combines the emotional impact of stories with the power of numbers (data) to illustrate the need for change. Social mathematics is a simple way to make data easier to grasp by relating it to things that we already understand. It is the practice that uses easy- to- visualize comparisons to make large numbers comprehensible and compelling by placing them in a social context that provides meaning. In other words, social mathematics is the mapping, graphing, and calculating social structure and social interaction to predict a rational and logical outcome.*

Business professionals dealing with the financial impact of risk and uncertainty are called actuaries, who provide assessments of financial security systems, with a focus on their complexity, their mathematics, and their mechanisms. The name of the corresponding discipline is actuarial science for which mathematics is essential. Codes using algebra and number theory, security and information compression technology are indispensable in practical implementation of mobile phones. Geometry is applied in the visualization of animation production and computer graphics, and conformational analysis of protein in life science.

For control technology needed to safely start, run, and stop automobiles, and practical implementation of the most environmentally friendly cars such as electric cars and hybrids, mathematical analysis, numerical analysis, and mathematical computation are essentially required. Besides, mathematics and mathematical sciences are useful in a variety of fields such as unraveling the mechanisms of traffic jams, telecommunications technology, virus displacement predictions, disease predictions and development of new medicines in genome data analysis, and high precision spam detection in search engines. One can see the reason behind mathematics being called the common language of science.

2.7 Mathematics Today

Mathematics is one of the oldest and most fundamental sciences. Today, Mathematics is all around us (as an essential tool), in everything we do. It is the building block for everything in our daily lives, including architecture (ancient and modern), art, finance, natural sciences (biology, chemistry, earth sciences, physics, space sciences, etc), formal sciences (computer science, statistics, etc), engineering, technology, medicine, agriculture, the social sciences (archeology, anthropology, economics, geography, psychology, sociology, etc), professional areas (agriculture, business, education, military sciences, environmental studies and forestry, etc) and even sports (in the analysis of sports performance, sports records, and strategy). Several engineering and medical problems find its solution through mathematics. Mathematical theory, computational techniques, algorithms, and the latest computer technology are used to solve economic, scientific, engineering, physics, and business problems. Thus, mathematics has become an exceedingly diverse subject over history, and there is a corresponding need to categorize the different areas of mathematics. Several different classification schemes have arisen; although they share some similarities, there are differences due to the different purposes they serve. Moreover, as mathematics develops, these classification schemes too must develop to account for newly established areas or newly discovered links between different areas. Classification is made more difficult by some subjects, often the most active, which bestride the boundary between different areas.

A conventional division of mathematics is into *pure mathematics* (or theoretical mathematics), mathematics studied for its intrinsic interest, and *applied mathematics*, mathematics which can be directly applied to real life problems.

Pure mathematics is the study of mathematics for its own sake, motivated for reasons other than application. It exhibits a trend towards increasing generality and abstraction. Theoretical mathematicians advance mathematical knowledge by developing new principles and recognizing previously unknown relationships between existing principles of mathematics. Although these mathematicians seek to increase basic knowledge without necessarily considering its practical use, such pure and abstract knowledge has been instrumental in producing or furthering many scientific and engineering achievements / advancements. Many theoretical mathematicians are employed as university faculty, dividing their time between teaching and conducting research.

The term *applied mathematics*, which loosely designates a wide range of fields with significant current use in the empirical sciences, includes numerical methods and computer science (which seeks concrete solutions, sometimes approximate) to explicit mathematical problems (e.g., differential equations, large systems of linear equations).

Applied mathematics has a major use in technology for modeling and simulation. For instance, the huge wind tunnels, formerly used to test expensive prototypes of aeroplanes, have all but disappeared. The entire design and testing process is now largely carried out by computer simulation, using mathematically tailored software. Applied mathematics also includes mathematical physics, which now strongly interacts with all of the principal areas of mathematics.

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Besides, probability theory and mathematical statistics are often considered parts of applied mathematics. Applied mathematicians use theories and techniques, such as mathematical modeling and computational methods, to formulate and solve practical problems in business, government, engineering, and the physical, life, and social sciences. For instance, they may analyze the most effective way to schedule airline routes between cities, the effects and safety of new drugs, the aerodynamic characteristics of an experimental automobile, or the cost-effectiveness of alternative manufacturing processes.

While working in industrial research and development and tackling difficult problem, applied mathematicians may develop or enhance mathematical methods. Some mathematicians, called cryptanalysts, analyze and decipher encryption systems (codes) designed to transmit military, political, financial, or law-enforcement-related information.

This broad division, however, is not always clear, i.e., these two classes are not sharply defined and they often overlap. Many subjects have been developed as pure mathematics to find unexpected applications later on. To develop accurate models for describing the real world problems, many applied mathematicians draw on tools and techniques that are often considered to be 'pure' mathematics. On the other hand, many pure mathematicians draw on natural and social phenomena as inspiration for their abstract research. Engineers use both pure and applied mathematics in the solution of problems. The distinction between pure and applied mathematics is now becoming less significant. Broad divisions, such as discrete mathematics and computational mathematics, have emerged more recently. Engineers use both pure and applied mathematics in the solution of problems.

Mathematics has many different branches (or areas), but some of them can be traced all the way back to antiquity (geometry, number theory), some others to baroque (analysis), some more to modernity (topology, non-Euclidean geometry), and still others have emerged only recently, in post-modern time (complexity theory, category theory). The foregoing classification is of course by no means complete.

The role of *probability theory*, *fuzzy set theory*, and *rough set theory* helping us in quantification and analysis of uncertainty in almost all branches of science, engineering, and humanities & social sciences is unquestionable.

Scientists set the hypotheses about various facts of nature and conduct experiments for their confirmation based on observations. S*tatistics* tests the hypotheses for their validity — it is the supreme judge; from its decisions there is no appeal. Paul Dirac said, "If there is a God, He is a great mathematician." Must be, otherwise how he will identify who will belong to what class 'hell' or 'heaven'. He will have to take help of Harold Hotelling or Prasanta Chandra Mahalanobish.

'Numerical Analysis' concerns the development of algorithms for solving all kinds of problems of continuous mathematics; it is a wide-ranging discipline having close connections with computer science, mathematics, engineering, and the sciences. So what if exact solution is not obtainable?

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Born during World War II, the discipline of *Operations Research* helps the decision makers from corporate managers to army generals make best possible operations, i.e., decisions in their own spheres of activities.

Computers not only operate on principles of mathematics, but are very useful in the often complex and time-consuming tasks mathematicians and scientists are required to perform. The operations and seemingly limitless abilities of computers are all based on the basic number system, namely, the binary number system. While binary is the basic numbering system of the computer, programmers work with different combinations of bytes to deal with computers at the most basic levels without the cumbersome methods required to program in binary. They use decimal (base 10), octal (base 8), and hexadecimal (base 16) notations, and of course, complex and very useful programming languages. No matter what the method used to program the computer, the basic operations are reduced to simple 1's and 0's in the end. But using many different combinations of these two numbers, manipulated in special ways, provides all the operations necessary for everything from computer games to complex scientific calculations.

In fact, in addition to the relatively standard fields of number theory, algebra, geometry, analysis (calculus), mathematical logic, and set theory, and more applied mathematics such as probability theory and statistics, the discipline of mathematics now covers a bewildering array of specialized areas and fields of study, including group theory, order theory, knot theory, sheaf theory, topology, differential geometry, fractal geometry, graph theory, functional analysis, complex analysis, singularity theory, catastrophe theory, chaos theory, measure theory, model theory, category theory, control theory, game theory, complexity theory, and many more.

2.8 Future of Mathematics

Mathematics is creative – it is exciting and multifaceted. Mathematics is the future – without mathematics modern key technologies will be unthinkable. Without mathematics, the entire universe would most likely remain a complete mystery to us. Among the many observable trends in mathematics, the most notable ones are: the subject is growing ever larger, the application of mathematics to bioinformatics is rapidly expanding, computers are ever more important and powerful, and the volume of data produced by science and industry for computer-facilitated analysis is explosively expanding.

Jules Henri Poincaré (1854-1912 CE), the French mathematician, theoretical physicist, engineer, and philosopher, wrote in 1908, "The true method of forecasting the future of mathematics lies in the study of its history and its present state". (Poincaré, 1908) But we lack mathematicians who study mathematical history. The historical approach can consist of the study of earlier predictions, and comparing them to the current ones to see how the predictions have fared, e.g., monitoring the progress of Hilbert's problems. (Yandell, 2002) A subject survey of mathematics itself, however, is now problematic – the mere expansion of the subject gives rise to the
issues of managing mathematical knowledge

Future of Mathematics

Mathematics is an absolutely critical part of our future – and we can maximize its impact for the public and private good over the next decades if we take the opportunity now. It is the multidisciplinary and universal nature of mathematics which makes this true: multidisciplinary because of its vast scope and universal because of the effectiveness of its processes. In some fields it plays a supportive role and in others, the lead. An example of a lead role which will be crucial to achieving the sort of economy we want is optimization of the processes in public and private sector enterprises.

Biology is a case in point – the slow uptake of mathematics and statistics in the university biology curriculum hampers our progress despite the demand for mathematically capable specialists at the research frontier. Thus, in our schools, we should connect mathematics and biology, the two disciplines which have not traditionally been close. Mathematics is meeting the biosciences in the 21st century much as mathematics met physics in the 20th.

Statistics plays an important role in every field of human activity such as determining the existing position of per capita income, unemployment, population growth rate, housing, schooling medical facilities etc in a country.

Today it holds a key position in almost every field like Industry, Commerce, Trade, Insurance, Physics, Chemistry, Economics, Mathematics, Biology, Botany, Medicine, Psychology, Astronomy, Natural and Social Sciences, etc; so application of statistics is very wide.

2.8.1 Experimental Mathematics

Experimental mathematics is a type of mathematical investigation in which numerical computation is used to investigate mathematical objects / structures and identify their properties and patterns. It is "that branch of mathematics that concerns itself ultimately with the codification and transmission of insights within the mathematical community through the use of experimental exploration of conjectures and more informal beliefs and a careful analysis of the data acquired in this pursuit." (Borwein, Borwein, Girgensohn, and Parnes, 1996, 2009) In other words, experimental mathematics is the use of computers to generate large data sets within which to automate the discovery of patterns which can then form the basis of conjectures and eventually new theory. As in experimental science, experimental mathematics can be used to make mathematical predictions which can then be validated or falsified based on additional computational experiments.

Mathematicians have always practised experimental mathematics. Historical records of early mathematics, for instance Babylonian mathematics, consist of lists of numerical examples illustrating algebraic identities. But, modern mathematics, beginning in the 17th century CE, evolved a tradition of publishing results in a final, formal, and abstract presentation; the numerical examples that may have led a mathematician to originally formulate a general theorem were not published, and were generally forgotten.

Mathematics is not, and never was, simply the end product of the search / research; the process of discovery is, and always was, an integral part of the subject. Carl Friedrich Gauss once wrote to his colleague Janos Bolyai (a Hungarian mathematician and a pioneer of non-Euclidean geometry), "It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment." The general idea of finding mathematical results by doing computational experiments has a distinguished, if not widely discussed, history. The method was extensively used, for example, by Carl Friedrich Gauss in the 1800s in his studies of number theory, and presumably by Srinivasa Ramanujan in the early 1900s in coming up with many algebraic identities. Evidently Gauss was an 'experimental mathematician' of the first order. For instance, analysis of the density of prime numbers by Gauss – when he was still a child – led him to formulate what is now known as the *Prime Number Theorem*, which was not proved conclusively until 1896, more than 100 years after its experimental discovery. The *Gibbs phenomenon* in Fourier analysis was noticed in 1898 on a mechanical computer constructed by Albert Michelson, a Polish-American physicist and Nobel laureate. Solitons were rediscovered in experiments done around 1954 on an early electronic computer by the Italian-American Nobel laureate physicist Enrico Fermi and his collaborators. (Wolfram, 2002) (In mathematics and physics, a *soliton* is a self-reinforcing solitary wave that maintains its shape while it travels at constant speed.) The chaos phenomenon was noted in a computer experiment by Edward Lorenz, the American mathematician and meteorologist, and a pioneer of chaos theory, in 1962. (Wolfram, 2002) A certain (erratic and apparently random) universal behavior known as chaos was discovered by the American mathematical physicist Mitchell Feigenbaum in iterated maps in 1975 by looking at examples from an electronic calculator. Using computer graphics, many aspects of fractals were found in the 1970s by the Polish-born French-American mathematician Benoit Mandelbrot, the father of fractals. Fractal geometry is considered as one of the major developments of 20th-century mathematics.

In the 1960s and 1970s, computer algebra was found to be used in a variety of algebraic identities, notably by Ralph William Gosper Jr. (b. 1943), also known as Bill Gosper, an American mathematician and programmer who (along with Richard Greenblatt) is considered to have founded the hacker community, and also holds a place of pride in the Lisp community. Starting in the mid-1970s computer algebra experiments were routinely done to find formulas in theoretical physics – though it was not mentioned while presenting the formulas. The idea that as a matter of principle there should be truths in mathematics that can only be reached by some form of inductive reasoning – like in natural science – was discussed by Kurt Gödel in the 1940s and by Gregory Chaitin in the 1970s. But it received little attention. With the release of *Mathematica* (a symbolic mathematics based computational software used in many scientific, engineering, mathematical, and computing fields) in 1988, mathematical experiments began to emerge as a standard element of practical mathematical pedagogy, and gradually also as an approach to be tried in at least some types of mathematical research, especially ones close to number theory. But even today, unlike all other branches of science, mainstream mathematics continues to be completely governed by theoretical rather than experimental methods. And even when experiments are done, their purpose is essentially always just to provide another way to look at traditional questions in traditional mathematical systems. "What I do … is, however, rather different: I use computer experiments to look at questions and systems that can be viewed as having a mathematical character, yet have never in the past been considered in any way by traditional mathematics." (Wolfram, 2002)

Experimental Mathematics

Experimental mathematics re-emerged as a separate area of study in the 20th century, when the invention of computer vastly increased the range of appropriate calculations, with a speed and precision far greater than anything available to earlier generations of mathematicians. A significant milestone and achievement of experimental mathematics was the discovery in 1995 of the Bailey–Borwein–Plouffe formula for the binary representation of the number Pi (π). This formula was discovered by numerical searches on a computer, not by formal reasoning; a rigorous proof found only afterwards. (Bailey, Borwein, Borwein, and Plouffe, 1997)

In their recent paper, *Experimental Mathematics: Recent Developments and Future Outlook*, David Bailey and Jonathan Borwein describe expected increases in computer capabilities as: better hardware in terms of speed and memory capacity; better software in terms of increasing sophistication of algorithms; more advanced visualization facilities; the mixing of numerical and symbolic methods. (Bailey, and Borwein, 2001)

The aims of experimental mathematics are "to generate understanding and insight; to generate and confirm or confront conjectures; and generally to make mathematics more tangible, lively and fun for both the professional researcher and the novice". (Borwein and Bailey, 2004, p. *vii*)

According to Jonathan Borwein and David Bailey, the uses of experimental mathematics mean the methodology of doing mathematics that includes the use of computation for: (1) gaining insight and intuition, (2) discovering new patterns and relationships, (3) using graphical displays to suggest underlying mathematical principles, (4) testing and especially falsifying conjectures, (5) exploring a possible result to see if it is worth formal proof, (6), suggesting approaches for formal proof, (7) replacing lengthy hand derivations with computer-based derivations, and (8) confirming analytically derived results. (Borwein and Bailey, 2004, p. 2)

Making use of numerical methods, experimental mathematics calculates approximate values for integrals and infinite series, and arbitrary precision arithmetic is often used to establish these values to a high degree of precision – typically 100 significant figures or more. Integer relation algorithms are then used to search for relations between these values and mathematical constants.

Working with high precision values reduces the possibility of mistaking a mathematical coincidence for a true relation. A formal proof of a conjectured relation will then be sought – it is often easier to find a formal proof once the form of a conjectured relation is known.

If a counterexample is being desired or a large-scale proof by exhaustion is being tried, distributed computing techniques may be used to divide the calculations between multiple computers.

General computer algebra systems, such as *Mathematica*, are now frequently used, although domain-specific software is also written for dealing with problems that require high efficiency. Experimental mathematics software usually includes error detection and correction mechanisms, integrity checks and redundant calculations designed to minimize the possibility of results being invalidated by a hardware or software error.

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"Before scientists can develop medicines or engineers can advance technology, they throw numbers onto whiteboards using concepts laid out by mathematicians sometimes centuries earlier. Unfortunately, mathematicians often get little recognition for their contributions to civilization and its history." Our deep obligation is due to all those great mathematicians who contributed so much to the development of civilization. Compilation of this work would not have been possible without taking help of the works of numerous scholars and authors, too many to be mentioned here; the authors acknowledge their profound indebtedness and gratitude to all of them. Our heartfelt thanks are due to the mathematicians of all eras and especially those whose works have been mentioned here, who contributed so much to the civilization of mankind through their dedicated work.

References

1. Bailey, David H.; Borwein, Jonathan M.; Borwein, Peter B.; Plouffe, Simon (1997 January), *The Quest for Pi*, The Mathematical Intelligencer, vol. 19, no.1, p. 50-57.
2. Bailey, David H. and Borwein, Jonathan M. (2001) *Experimental Mathematics: Recent Developments and Future Outlook*, Mathematics unlimited: 2001 and beyond, Springer, CiteSeerX 10.1.1.138.1705.
3. Borwein, J.; Borwein, P.; Girgensohn, R.; Parnes, S. (1996), *Making sense of experimental mathematics*, The Mathematical Intelligencer, 18(4), p. 12-18.
4. Borwein, J. and Bailey, D. (2004), *Mathematics by Experiment: Plausible Reasoning in the 21st Century*, A.K. Peters, Massachusetts, USA, p. vii. ISBN 1-56881-211-6.
5. Borwein, J.; Borwein, P.; Girgensohn, R.; and Parnes, S. (2009), *Experimental Mathematics: A Discussion*, retrieved 27-10-2009.
6. Boyer, Carl B. (1991a), *The Arabic Hegemony* in *A History of Mathematics*, Second Edition, revised by Uta C. Merzbach, John Wiley & Sons, Inc., New York, USA, p. 239, ISBN 0471543977.
7. Boyer, Carl B. (1991b), *The Arabic Hegemony,* in *A History of Mathematics*, Second Edition, revised by Uta C. Merzbach, John Wiley & Sons, New York, ISBN 0-471-54397-7.
8. Boyer, Carl B. (1991c), *China and India*, in *A History of Mathematics*, revised by Uta C. Merzbach, John Wiley & Sons, Inc., New York, ISBN 0471543977.
9. Chisholm, Hugh, ed. (1911), *Albategnius*, Encyclopædia Britannica (11th ed), Cambridge University Press, New York.
10. Coolidge, J.L. (1949 March), *The Story of the Binomial Theorem*, Amer. Math. Monthly, Vol. 56, No. 3, p. 147-157.
11. Davis, Philip J. and China, Phillip J. (1985), *3.1416 and All That*, Birkhauser Boston, Boston, Massachusetts, USA.
12. Davis, Philip J. and Hersh, Reuben (1981), *The Mathematical Experience*, Birkhauser Boston, Massachusetts, USA; reprinted by Houghton Mifflin Harcourt, Boston, Massachusetts, USA, 1998, ISBN 0395929687.
13. Devlin, Keith (1998), *Goodbye, Descartes: The End of Logic and the Search for a New Cosmology of the Mind*, John Wiley & Sons, ISBN 0471251860.
14. Devlin, Keith (2011), *Mathematics Education for a New Era: Video Games as a Medium for Learning*, A K Peters/CRC Press, Massachusetts, USA, ISBN 1568814313.
15. Divakaran, P.P. (2007), *The first textbook of calculus: Yukti-bhaṣa*, Journal of Indian Philosophy 35, p. 417-433.
16. Ernest, Paul (1991), *The Philosophy of Mathematics Education*, Falmer Press, London.
17. Ernest, Paul (1998), *Social Constructivism as a Philosophy of Mathematics*, State University of New York Press, Albany, USA.
18. Gandz, Saloman (1936), *The sources of Khwarizmi's algebra*, Osiris 1, p. 263-77.
19. Goonatilake, Susantha (1998), *Toward a Global Science: Mining Civilizational Knowledge*, Indiana University Press, Indiana, USA, ISBN 0-253-33388-1.
20. Grattan-Guinness, Ivor (1997), *The Rainbow of Mathematics: A History of the Mathematical Sciences*, W.W. Norton, New York City, ISBN 0-393-32030-8.
21. Gray, Jeremy (2008), *Plato's Ghost: The Modernist Transformation of Mathematics*, Princeton University Press, Princeton, USA.
22. Hawking, Stephen (1998), *A Brief History of Time*, Bantam Dell Pub Group, Westminster, Maryland, USA, ISBN 0553380168.
23. Himmelfarb, Gertrude (1995 September), *Academic Advocates*, Commentary, Vol. 100, No. 3, p. 47.

References

24. Kangshen, Shen; Crossley, John N.; Lun, Anthony W.C. (Editors) (1999), *The Nine Chapters on the Mathematical Art: Companion and Commentary*, Oxford University Press, Oxford, UK, ISBN 0198539363.
25. Katz, Victor J. (1995), *Ideas of Calculus in Islam and India*, Mathematics Magazine 68 (3): 163-74.Katz, Victor (1998), *A History of Mathematics: An Introduction* (2 ed.), Addison-Wesley, Reading, Massachusetts, p.879, ISBN 0-321-01618-1.
26. Katz, Victor (1998), *A History of Mathematics: An Introduction* (2 ed.), Addison-Wesley, Reading, Massachusetts, p. 879, ISBN 0-321-01618-1.
27. Kline, Morris (1953), *Mathematics in Western Culture*, Pelican, Great Britain, p. 150-151.
28. Kulkarni, R.P. (1978), *The Value of π known to Sulbasutras*, Indian Journal for the History of Science, vol. 13, no. 1, p. 32-41.
29. Lakatos, Imre (1976), *Proofs and Refutations: The Logic of Mathematical Discovery*, Cambridge University Press, Cambridge, UK.
30. Lakatos, I. (1978), *Philosophical Papers*, Vol.2, Cambridge University Press, Cambridge, UK.
31. Livio, Mario (2010), *Is God A Mathematician?*, Simon & Schuster, London, ISBN 0743294068.
32. MacKenzie, D. (1993), *Negotiating arithmetic, constructing proof: the sociology of mathematics and information technology*, Social Studies of Science: 23, p. 37-65.
33. Majumdar, P.K. (1981), *A rationale of Brahmagupta's method of solving ax + c = by*, Indian Journal of History of Science, vol. 16, no. 2, p. 111-117.
34. Moslehian, Mohammad Sal (2005 November), *Post Modern View of Humanistic Mathematics*, Resonance, p. 98-105.
35. O'Connor, J.J. and Robertson, E.F. (1999), *Omar Khayyam*, MacTutor History of Mathematics archive, School of Mathematics and Statistics, University of St Andrews, Scotland.
36. O'Connor, J.J. and Robertson, E.F. (2005), *Abu'l Hasan ibn Ali al-Qalasadi*, MacTutor History of Mathematics archive, School of Mathematics and Statistics, University of St Andrews, Scotland.
37. Plofker, Kim (2009), *Mathematics in India: 500 BCE – 1800 CE*, Princeton University Press, Princeton, New Jersey, USA, ISBN 0-691-12067-6.
38. Poincare, Henri (1908), *The Future of Mathematics*, translation of the French original: *L'avenir des mathématiques*, in *Revue générale des sciences pures et appliquées* 19 (1908), p. 930-939; also appeared in: *Circolo Matematico di Palermo*; *Bulletin des Sciences mathématiques*; *Scientia*; and *Atti del IV° Congresse internazionale dei Matematici*, lecture held at the Eighth International Congress of Mathematicians, Rome, Italy, 1908.
39. Puttaswamy, T.K. (2000), *The Accomplishments of Ancient Indian Mathematicians*, p. 411-2, in Selin, Helaine and D'Ambrosio, Ubiratan, eds.. *Mathematics Across Cultures: The History of Non-western Mathematics*, Springer, New York City, ISBN 1-4020-0260-2.
40. Raloff, Janet (1996), *When Science and Beliefs Collide*, Science News, Vol. 149, No. 23, p. 360-364.
41. Rashed, R. and Armstrong, Angela (1994), *The Development of Arabic Mathematics*, Springer, New York City, p. 11-12, ISBN 0-7923-2565-6.
42. Rashed, R. and Vahabzadeh, B. (2000), ***Omar Khayyam, the Mathematician***, Bibliotheca Persica Press, New York.
43. Rosen, Fredrick (1831), *The Algebra of Mohammed Ben Musa*, Kessinger Publishing, Kila, Montana, USA, ISBN 1-4179-4914-7.
44. Russell, Bertrand (1907 November), *The Study of Mathematics*, The New Quarterly, I, p. 29-44; Cambridge University, Cambridge, UK, 1910.
45. Russell, Bertrand (1967), *Autobiography*, George Allen and Unwin Ltd, UK, V1, p. 158.
46. Smith, Wallace G. (2013), *Our Mathematical Universe*, Tomorrow's World, July-August issue.
47. Whitting, C.E.J. (translator) (1947), *Al Fakhri On the Systems of Government and the Moslem Dynasties*, Luzac & Co., London.
48. Wigner, Eugene P. (1960), *The Unreasonable Effectiveness of Mathematics in the Natural Sciences,* Communications on Pure and Applied Mathematics, vol. 13, no. 1, p. 1-14.
49. Woepcke, F. (1853), *Extrait du Fakhri, traité d'Algèbre par Abou Bekr Mohammed Ben Alhacan Alkarkhi, Imprimerie Impériale,* Paris; reprinted in Hildesheim in *1982,*.
50. Wolfram, Stephen (2002), *A New Kind of Science*, Wolfram Media, Champaign, Illinois, USA, ISBN 1-57955-008-8, p. 899.
51. Yadegari, M. (1980), *The Binomial Theorem: A Widespread Concept in Medieval Islamic Tradition*, Historia Mathematica, vol. 7, p. 401-406.
52. Yandell, Benjamin H. (2002), *The Honors Class: Hilbert's Problems and Their Solvers*, A K Peters Ltd., Natick, Massachusetts, USA, ISBN 1568811411.
53. Youschkevitch, A., and Rosenfeld, B. (1973), *al-Khayyami*, in Dictionary of Scientific Biography (Volume 7), C. Gillispie (ed.), Charles Scribner's Sons, New York, p. 323–334.
54. Zawadowski, Waclaw M. (1990), *Postmodernism and Postmodern Mathematics in Schools*, Unpublished paper contributed to the Mathematics Education into the 21st Century Project.