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The Mathematics and Physics Behind Piano Tuning

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Abstract:

A piano can never be perfectly tuned like other instruments using harmonics because it has too many strings. This paper is an exploration of how sound is produced, and a discussion of a widely used solution to the piano tuning problem, the Equal Tempered Tuning. The equal tempered keyboard is a method of tuning used to tune pianos as accurately as possible. In this method, the frequency of a key is determined by multiplying the frequency of the previous key by the 12^{th} root of 2, $\sqrt[12]{2}$.

Keywords: Piano Tuning, Equal Tempered Keyboard, Sound, Waves, Harmonics.

1. Introduction:

Music and mathematics overlap in a lot of different ways and have a relationship which goes back thousands of years to the very origins of music itself. One of the most common similarities between music and mathematics are the use of patterns in music. Patterns and numbers are used to teach music every day. Mathematics is used in music in more ways than one-

- Reading Music: Reading music involves measures, beats, time signatures, note values etc. All of these inculcate a concept of mathematics in one way or another. A musical piece is divided into sections, called measures, and each measure has an equal number of beats. A time signature defines the amount of beats in one measure. Note values refer to the amount of beats a particular note has to be held down for.
- Frequency of the notes: The frequencies of the notes in music are also related to mathematics. Pythagoras found out through research that different weights and vibrations of strings make different sounds. Through this discovery we now know that the pitch of a vibrating string is directly proportional to the length of the string.
- Patterns: Patterns exist throughout music. Be it through rhythmic passages, rest, harmonies, etc. The structure of a song, or musical piece, is a pattern in itself. For example, a common song has repeating choruses preceded by verses.

2. Literature Review:

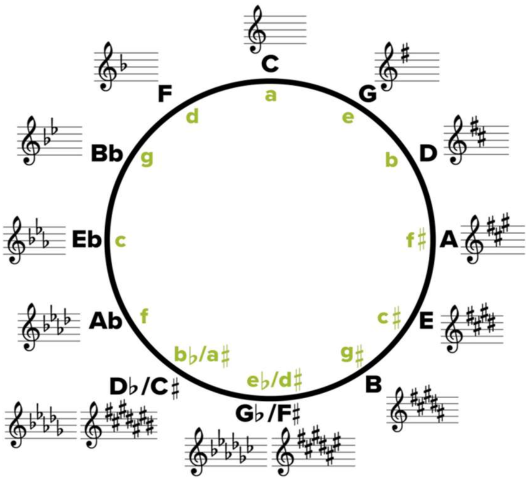
This paper is an extension of a study conducted on the mathematics behind piano tuning by

Tony Phillips at Stony Brook University (2002)[1]. This study is based on the specific example of the Hermit Thrush and how natural harmonics could be incorporated into the tuning of a piano. This current paper explores this problem further and expands the problem by talking about more than just natural harmonics. In addition to the tuning of a piano, this paper talks about how sound is created and controlled to behave in a certain way. The study conducted by Tony Phillips brings up the equal tempered keyboard and then talks about the incorporation of natural harmonics using the wave equation. This was not relevant to the aim of this paper and since this paper was broader, it only discussed the equal tempered keyboard.

2.1 The Circle of Fifths:

The circle of fifths is one of the most important concepts one learns in music since it helps in understanding notation, transposition, key signatures, and the general structure of music. The circle of fifths visually shows the relationship among the 12 tons of the chromatic scale, their respective key signatures and their major and minor keys.

Figure 1: Circle of Fifths[2]



The outer circle shows the key signatures while the uppercase and lowercase letters signify the major and minor keys respectively. In the figure above, when looked at clockwise, the major keys are separated by an interval of 5ths. For example, G is a 5th up from C, D is a 5th up from G, and so on. The same pattern is repeated for the minor keys, which are shown in the inner circle. Another noticeable pattern is the number of sharps and flats as one looks at the circle of fifths clockwise and counter-clockwise.

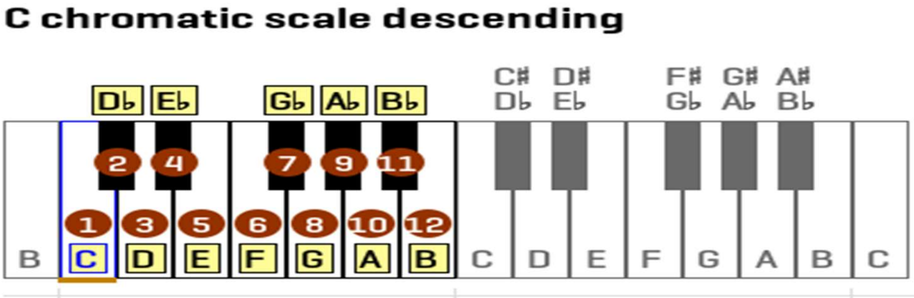
[1] Piano1. (2000). E. http://www.math.stonybrook.edu/%7Etony/whatsnew/column/piano-0900/piano1.html

[2] *Circle of Fifths*. (2021, November 9). [Illustration]. Circle of Fifths Guide: Why and How Is It Used? https://www.musicnotes.com/now/tips/circle-of-fifths-guide/

2.2 The Mathematics behind the Circle of Fifths:

An observation that can be made about the circle of fifths is that it is full of patterns. Some of them stand out and are plainly visible to the naked eye, while others require a little more attention to detail. In this section, some of those patterns are going to be addressed.

Figure 2: Chromatic Scale on Piano with numbered notes[3]



In musical terminology, a fourth is 5 chromatic steps, which on a piano would be 5 total white keys and black keys. Similarly, a fifth is 7 chromatic steps. These two numbers, 5 and 7, are relatively prime to 12, which means that they don't share factors of 12 (there are 12 notes in the chromatic scale, i.e., 12 total white and black keys form a chromatic scale). The numbers that are relatively prime to 12 are 1, 5, 7, and 11. 5 chromatic steps and 7 chromatic steps form a fourth and a fifth as mentioned above, and going up by 1 step or 11 steps produces the same sequence of notes in different octaves. The numbers that are not relatively prime to 12, or the numbers that do share factors of 12, are 2, 3, 4, 6, 8, 9, and 10. Going up by 2 chromatic steps forms a whole-tone scale, for example, C to D, or B to C#. Going up 10 chromatic steps would produce the same set of notes, just in different octaves. Similarly, going up 3 or 9 chromatic steps produces a diminished chord, 4 or 8 chromatic steps produce an augmented chord, and 6 steps forms a tritone, or an augmented fourth.

Using this knowledge of the chromatic scale, one can imagine a non-traditional scale that divides the octave into anything other than 12 parts. Assuming this new scale has *n* notes, going over the notes in steps of size *m* will cover all *n* notes only if *m* and *n* are relatively prime. For instance, imagine a scale which is divided into 15 parts, or notes. To cover all 15 notes, we could go up 4 steps at a time, playing the notes 1, 5, 9, 13, 2, 6, 10, 14, 3, 7, 11 and 15. In this case, 4 steps after 13 will lead to 17, but as mentioned earlier, the octave has 15 notes making 17 note number 2 in the octave above.

If *n* (the number of parts the octave is divided into) and *m* (the size of the steps to cover the octave) are not relatively prime, assume *d* to be their greatest common divisor. This way, going up *d* parts at a time would mean cycling through „ notes making *d* distinct cycles. Using the same 15 note scale example above, going up in intervals of 10 would cover 3 different notes. With these notes, 5 different three-note chords could be made. For example, one of the chords would consist of the notes 1, 11 and 6. If a scale had a prime number of notes, then every interval (other than an octave) would cycle through all notes.

[3] C Chromatic Scale. (2021, November 9). [Illustration]. C Chromatic Scale. https://m.basicmusictheory.com/c-chromatic-scale

3. Aim of the Study:

The aim of this paper is to explain the physics behind how sound is produced, introduce the tuning problem, explain and discuss the potential solution of the equal tempered keyboard, and evaluate it against the more accurate but less plausible tuning method of using harmonics.

4. Methodology:

This paper is an extension of a study conducted on the mathematics behind piano tuning by Tony Phillips at Stony Brook University (2002). This study is based on the specific example of the Hermit Thrush and how natural harmonics could be incorporated into the tuning of a piano. This current paper explores this problem further and expands the problem by talking about more than just natural harmonics. In addition to the tuning of a piano, this paper talks about how sound is created and controlled to behave in a certain way. The study conducted by Tony Phillips brings up the equal tempered keyboard and then talks about the incorporation of natural harmonics using the wave equation. This was not relevant to the aim of this paper and since this paper was broader, it only discussed the equal tempered keyboard.

This paper is written for a target audience of people who understand music theory and physics. Even though most of the concepts talked about in this paper are simplified and explained from scratch, to successfully understand this paper, one must have a rudimentary understanding of the piano, mathematics and physics. To write this paper, each individual concept had to be explained, like the basics of music theory in terms of scales, note intervals, how sound is generated, frequencies, notes, etc. To gather information about these concepts, different research papers helped with credible and valuable information. For information related to music theory, music journals and theory books helped and for the basics of the physics concepts, websites like Britannica provided all the necessary information. The paper is structured in a way that all the background information is given chronologically, explaining all the concepts required to understand the final section of the paper, which talks about the solution to the problem incorporating concepts of physics, mathematics, and music theory.

5. Standing Waves:

When two identical waves move in opposite directions along a line, a standing wave is formed which doesn't travel through space of along a string. It is sinusoidal, just like the two waves that it consists of, and it oscillates at the same frequency. Stretching a rubber band between two fixed points, pulling it from the middle, and releasing it so that it vibrates in position is a way of visualizing the standing wave. In musical instruments, a standing wave is created when the oscillating medium, like the strings of a guitar, is driven at one end. This generates the standing wave not by the two component waves, but by the reflections of the original wave off the ends of its vibrating system.

5.1. Stretched Strings:

When a string stretched between two fixed points its plucked, it vibrates and creates a sound. Some of the earliest sources of music were from stretched strings. Examples of musical instruments that use stretched strings are guitars, violins, cellos etc.

5.2. Harmonics and Fundamentals[4]:

The equation for the speed (v) of a wave in a stretched string of a given mass per unit length (μ) experiencing a tension (F) is: $\boldsymbol{v = \sqrt{\frac{F}{\mu}}}$

The wavelength (λ) of the vibration produced when a string of a given length is plucked in the middle is twice the length (L) of the string: $\boldsymbol{\lambda = 2L}$

The frequency (f_1) of this vibration can then be obtained by the following $\boldsymbol{f_1 = \frac{v}{\lambda} = \left(\frac{1}{2L}\right)\sqrt{\frac{F}{\mu}}}$

The vibration that has the lowest frequency for a given type and length of string experiencing a given amount of tension is known as the fundamental, or the first harmonic. Standing waves can also be created in stretched strings, as shown in Figure 3. Figure 3 is a graphical representation of standing waves in stretched strings. As can be seen, they are all sinusoidal curves. The first curve, labelled n = 1, is the fundamental, or the first harmonic. A string can only be stretched if it is fixed on both ends. These ends are attached to fixed points such that there is no motion of those points. These points are called nodes, or nodal points, and are represented by the letter *N* on the graphical representation in Figure 3. When the string is pulled and released, it oscillates. These extreme positions of these oscillations are shown by the dotted and solid curves. In the center of the string, where the amplitude of the curve is the greatest, or in other words, the vertex of the graph shown in Figure 3, is the antinode, or the antinodal point represented by the letter *A*.

Figure 3: Standing Waves in stretched strings[5]

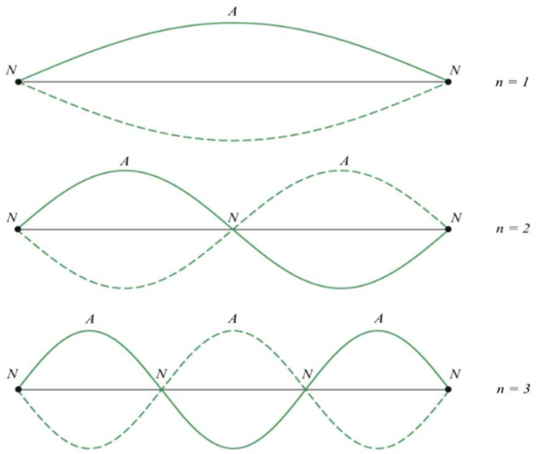


Figure 3 also shows the next two vibrational modes of the string where n = 2 and n = 3. The strings are divided into equal parts called loops, each of which is half a wavelength long. The relationship between the wavelength and the length of the string is shown by the equation:

[4] *sound - Standing waves*. (n.d.). Encyclopedia Britannica. https://www.britannica.com/science/sound-physics/Standing-waves
[5] *sound - Standing waves*. (n.d.). Encyclopedia Britannica. https://www.britannica.com/science/sound-physics/Standing-waves

$$\lambda_n = \frac{2L}{n}$$

In this equation, *n* is the number of loops in the standing wave. From the equation above, it can be deduced that the frequencies of the vibrations (fn) are represented by: $f_n = \frac{v}{\lambda_n} = \left(\frac{n}{2L}\right)\sqrt{\frac{F}{\mu}}$

or, in terms of the fundamental frequency $f1$: $f_n = nf_1$

Here *n* is the harmonic number since the sequence of frequencies are harmonics, or integral multiples, of the fundamental frequency, or the first harmonic. As can be seen in Figure 3, the second representation which is labelled n = 2, which is called the second harmonic, shows the string vibrating in two sections making the string a full wavelength long. Since the wavelength of the second harmonic, labelled n = 2, is half of the wavelength of the fundamental, labelled n = 1, the frequency of the second harmonic is twice that of the fundamental. Similarly, the frequency of the third harmonic, labelled n = 3, is thrice that of the fundamental.

5.1.2. Overtones:

Overtones are another term used for standing waves. The fundamental is referred to as the first overtone, the first harmonic is referred to as the second overtone, the second harmonic is referred to as the third overtone, and so on. The term 'overtone' is generally used when referring to standing waves with higher frequencies. Harmonics, on the other hand, are integral multiples of the frequency of the fundamental. Overtones and harmonics are sometimes also called resonances. When a system is said to resonate, it vibrates at a larger amplitude since when the system vibrates at a certain frequency, it is subjected to external vibrations of the same frequency. The overtone series, which is the sequence of frequencies defined by the equation,

$$\boldsymbol{f_n = nf_1}$$

plays a large part in the analysis of musical tone quality and musical instruments. For example, if the fundamental frequency is the note G2, the first 10 frequencies, or harmonics, in the overtone series will be close to the notes G3, D4, G4, B4, D5, F5, G5, A5, and B5. The frequencies of the octaves (harmonics 1, 2, 4, and 8), which correspond to the notes G2, G3, G4, and G5, are the same as the frequencies of the notes on the equal tempered keyboard. However, the rest of the notes in the series differ slightly from the notes on the equal tempered scale by being either slightly sharp or flat. The 7th harmonic, or the note F5, is out of tuned as compared to the note on the equal tempered keyboard. In Europe, during the Middle Ages, keyboard instruments were tuned using a method called just intonation. In this method, the primary chords were true frequencies of the overtone series, and the notes in the chord were members of a single overtone series.

5.1.3. Mersenne's Laws[6]:

Mersenne's Laws are three derived laws or statements that show the relationship between the length, tension and mass per unit length of the string, and the fundamental frequency of the string when it is stretched. These statements can be written as follows:

1. The fundamental frequency of a stretched string is inversely proportional to the length of the string, keeping the tension and the mass per unit length of the string constant: $\boldsymbol{f_1 \propto \frac{1}{L}}$
2. The fundamental frequency of a stretched string is directly proportional to the square root of the tension in the string, keeping the length and the mass per unit length of the string constant: $f_1 \propto \sqrt{F}$
3. The fundamental frequency of a stretched string is inversely proportional to the square root of the mass per unit length of the string, keeping the length and the tension in the string constant: $\boldsymbol{f_1 \propto \frac{1}{\sqrt{\mu}}}$

These laws explain how string instruments are built and how they work. For example, to make the tension in each guitar string almost the same to get a more uniform sound, the lower strings of the guitar are made with a greater mass per unit length, which means they are heavier and thicker, and the higher strings are lighter and thinner. Another example is a grand piano. In a grand piano the tension in each string is about 100 pounds, leading to a total force of about 40,000 to 60,000 pounds on the frame. Since a large variation in the tension applied to the higher and lower strings could lead to the warping of the piano frame, the higher strings are shorter and thinner while the lower bass strings are longer and far thicker. The wires in a grand piano are stiff due to this construction, which makes the overtones a little higher and makes the frequency deviate slightly from the ideal, forming a very important characteristic of the sound of a piano.

6. Natural Harmony and the Tuning Problem:

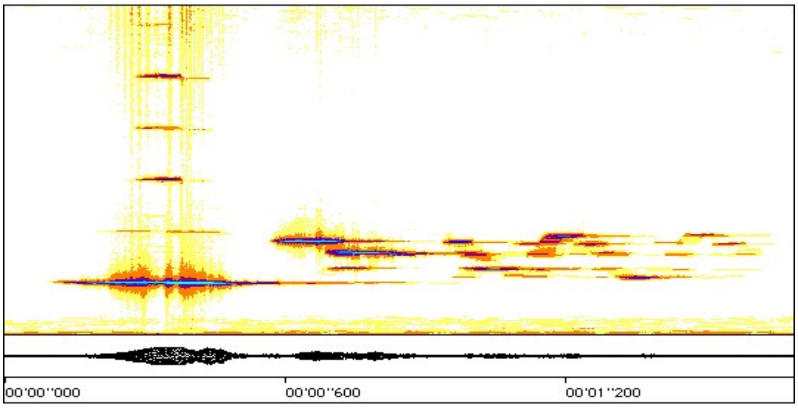
Every stretched chord or open pipe has a frequency, which is its "note", that is dependent mostly on the length, tension and density of the chord or the length of the pipe. This note is also referred to as the fundamental. Along with this fundamental, a string or a pipe has a series of higher frequency vibrations, or higher notes. A chord may vibrate as two cords of half its length joined end-to-end, or three cords of one-third the length, or four, and similarly for the pipe. Since the frequency of the vibration is inversely proportional to the length of the chord, with all else

[6] *sound - Standing waves*. (n.d.). Encyclopedia Britannica. https://www.britannica.com/science/sound-physics/Standing-waves

constant, these higher frequencies, or notes, are called the higher harmonics of the fundamental.

A natural harmony in musical terms is a harmony that exists in nature or is created by the forces of nature, as opposed to having been created by man. An example of a natural harmony from nature is the Hermit Thrush, a bird that is fairly common in eastern North America. The Hermit Thrush sings phrases in which there is a long note followed by a sequence of more rapid ones.

Figure 4: Sonogram of a Hermit Thrush[7]



This is the sonogram of one such bird singing which plots the frequency of a note against the time. As can be seen, there is a single long note followed by faster notes. The first long note shows the higher harmonics. The frequencies of the notes are 3531, which is the fundamental[8], 6976, 10336, 13953, 17399 and 20758 cycles per second. These frequencies correspond to the notes A7, A8, E9, A9, C#10, E10 respectively in standard notation, where A4 = 440 cycles per second. In the notes written above, the alphabets signify the note, and the numbers after the notes signify the octave that the notes belong to on a piano.

Higher harmonics give a set of pitches that are directly related to the fundamental. The incorporation of these natural harmonics in a musical scale is the basic mathematical problem in music. Through this research paper, an approximate solution called the Equal Temperament, will be spoken about. This solution is used in the tuning of pianos, and hence this mathematical problem is referred to as the tuning problem.

7. Frequencies and Notes:

[7] Sonogram. (2000). The Mathematics of Piano Tuning. http://www.math.stonybrook.edu/~tony/whatsnew/column/piano-0900/piano1.html

[8] *piano1*. (2000). E. http://www.math.stonybrook.edu/%7Etony/whatsnew/column/piano-0900/piano1.html

The frequency spectrum, or the frequency of a sound is physically observable. The distance between two pitches, in other words the interval between two sounds, can be represented as a function of the ratio of the frequencies of those sounds. Two sounds with frequencies in a 2:1 ratio is judged to be the same note in different pitches, or octaves. In an analysis of notes, sounds should be arranged on the basis of the logarithms of their frequencies because logarithm functions have the property, $log\,\frac{a}{b} = log(a) - log(b);$ i.e., the distance between the logarithms of the two sounds depends on the ratio of their arguments. In a geometric representation of the set of pitch classes, frequencies that differ by a factor of 2 should correspond to the same point, since they belong to the same pitch class. The simplest way to do this is to use logarithms to the base 2, and to only keep the fractional part of the logarithm. This way if-

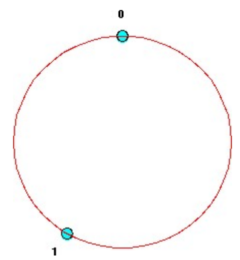
$$a = 2b$$
$$log_2 a = log_2 2 + log_2 b$$
$$\therefore log_2 a = 1 + log_2 b$$

making the fractional parts the same.

This way, a geometric representation of a set of pitch classes is a circle of radius 1. The best way to visualize the geometric picture is to take the pitch class corresponding to the frequencies which are exact powers of 2 as the basepoint. For example, 256 (approximately middle C), 512, 1024 etc. Then each other pitch class gets displayed along the circle at a distance from the basepoint equal to the fractional part of its logarithm to the base 2. As the frequency approaches twice the base frequency, the fractional part of its logarithm to the base 2 gets closer to 1, and we get back to the basepoint, as required.

To the right is a circle of length 1 with a basepoint marked “0”. The pitch class corresponding to its third harmonic, or its second overtone is marked “1”. The distance between the two points on the circle (clockwise) is the fractional part of the logarithm of the ratio of their frequencies, which in this case is the fractional part of log#3, or 0.584963.

Figure-5 : Circle with Length 1 and Basement 0[9]



The pitch class "1" has its own third harmonic, or second overtone added to the collection; pitch class "2." It will be placed on the circle at distance 0.584963 clockwise from pitch "1". This

[9] *piano1*. (2000). E. http://www.math.stonybrook.edu/%7Etony/whatsnew/column/piano-0900/piano1.html

process is repeated to give pitch classes "3", "4", "5" etc. Each of these classes is naturally related to the one before.

8. The Equal-Tempered Keyboard:

In equal temperament, each semitone is measured in relation to the previous one; each semitone's frequency increases by a factor of the 12th root of 2. This is an irrational number rounded off to about 1.059… since, as mentioned earlier, when multiplied by the whole number harmonics or overtones, there is a slight inharmonicity in the frequencies of the notes. When each semitone is multiplied by the 12th root of 2 to get the next semitone, eventually each octave ends up having a multiplier of 2, which means that if the note A in a particular octave was measured at 440 hertz, the octave below would have a frequency of 220 hertz, and the octave above would have a frequency of 880 hertz. Equal temperament is called a divisional system since it involved dividing the octave into 12 semitones.

In the past, alternative European tuning systems, like just intonation introduced earlier, were cyclic systems. This means that the frequencies were calculated in intervals by adding other pure intervals. This method leads to multiple intonational differences when the calculations reach a key that is distantly related (usually with more sharps and flats) to the one they started on and sounds quite out of tune when related to instruments with fixed intonation. To put it simply, if a song played in C major, which has no sharps or flats, is transposed to B major, which has 5 sharps, the song would sound wrong because the intervals are different. In equal temperament, however, these small intonational defects are accounted for and equally divided in the octave making only the octave the acoustically pure interval.

9. Conclusion:

Most instruments produce sound through strings or through air. All of these instruments can be mathematically tuned using harmonics, for example on a violin, which has only 4 strings, the 2nd harmonic of the first string, or the G string, is equal to the 3rd harmonic of the second string, or the D string. Similarly, in a guitar, which has 6 strings, the 4th harmonic of the first string, the E string, is equal to the third harmonic of the second string, the A string. A piano, however, has too many strings to be tuned using harmonics. It has one string for each of the 12 semitones of the western chromatic scale, and with the number of keys, the total number of strings comes up to about 230. Tuning the piano using harmonics would lead to inaccurate frequencies and the octave wouldn't be directly multiplied by 2. The equal tempered keyboard offers a compromise by using an irrational number to multiply each frequency since a whole number does not yield accurate results. This makes it piano tuning as accurate as it gets but still leaves some of the notes either slightly sharp or flat. The equal tempered keyboard is nowhere near as accurate as harmonic tuning is for other instruments, but it is the most viable compromise.

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